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Exam / Tentamen
Set Theory & Metamathematics
MM7048, 7,5 hp, VT2026
Tuesday 2026-05-26

This exam consists of 4 questions, worth a total of 40 points. Note that some subproblems are intentionally quite challenging, and you do not need to complete all questions in order to achieve a good grade. You may submit your answers in either English or Swedish. Write clearly and motivate your answers carefully.
Good luck! — Lycka till!

1. **Ranks and choice.** For this question, work just in ZF.

(a) Recall the Kuratowski definition of ordered pairs, $(x, y) \coloneqq \{\{x, y\}, \{x\}\}$. Give the rank of (x, y) in terms of the ranks of x and y.

For any set X, let $\mathrm{AC}[X]$ be the axiom of choice for subsets of X, in the form "for any set $\mathfrak{Y}$ of non-empty subsets of X, there exists some choice function f defined on $\mathfrak{Y}$; that is, $\operatorname{dom} f = \mathfrak{Y}$, and $f(Y) \in Y$ for each $Y \in \mathfrak{Y}$".

(b) Show that $\mathrm{AC}[\mathbb{R}]$ is equivalent to $(\mathrm{AC}[\mathbb{R}])^{V_\omega}$.

(c) Give some ordinal α and set $X \in V_\alpha$ such that $\mathrm{AC}[X]$ is not equivalent to $\mathrm{AC}[X]^{V_\alpha}$.

2. **Well-founded and ill-founded models of set theory.** Work in ZFC. Say that a structure $\mathcal{M} = (M, \varepsilon)$ for the language of set theory is *well-founded* just if ε is well-founded in the usual sense: that is, if every non-empty $X \subseteq M$ has some ε-minimal element, or equivalently, if there is no infinite $\ni$-descending sequence $x_0 \ni x_1 \ni x_2 \ni \ldots$ Conversely, call a structure *ill-founded* if it is not well-founded.

(a) Show that any *standard* structure $(N, \in)$ (i.e. with actual membership $\in$ as the relation) is well-founded.

(b) Show that if $\mathcal{M} = (M, \varepsilon)$ is a well-founded model of ZF, then it is isomorphic to some transitive standard model $(N, \in)$.

(c) Show that if $\mathcal{M}$ is a well-founded model of set theory, then $[\![\ (\omega, <)\ ]\!]^{\mathcal{M}} \cong (\omega, <)$ — that is, "$\mathcal{M}$ has standard arithmetic". (Here $[\![\ (\omega, <)\ ]\!]^{\mathcal{M}}$ means the interpretation in $\mathcal{M}$ of the usual ZF-definition of ω, together with its usual ordering.)

(d) Show that if ZF is consistent, then there exists some ill-founded model of ZF. *(Hint: recall the compactness theorem for models.)*

(e) *Optional, not for credit: discuss why this doesn't contradict the fact that in any model of ZF, the foundation axiom certainly holds!*

3. **Cardinality calculations for the constructible universe.**

Work in ZFC. Recall the "definable power set" construction $\mathcal{D}(X)$, used in the definition of $\mathbf{L}_{\alpha+1} \coloneqq \mathcal{D}(\mathbf{L}_\alpha)$.

(a) Give the definition of $\mathcal{D}(X)$. *(We and Kunen considered a couple of different equivalent definitions of $\mathcal{D}$; give whichever you prefer here.)*

(b) Show that $|\mathcal{D}(X)| = |X|$ whenever $|X| \geq \aleph_0$.

(c) Show that for each ordinal α, $\alpha \subseteq \mathbf{L}_\alpha$. *(Note any facts about absoluteness you use.)*

(d) Show that for each ordinal $\alpha \geq \omega$, $|\mathbf{L}_\alpha| = |\alpha|$.

(e) Show that assuming $\mathbf{V} = \mathbf{L}$, the $\mathbf{L}$-rank $\mathrm{rk}_{\mathbf{L}}(x)$ of a set may differ from its ordinary rank $\mathrm{rk}(x)$.

4. **Concrete calculations in forcing.** Throughout, let $\mathcal{M}$ be a countable transitive model of set theory, and work with the poset $\mathbb{P} = \mathrm{Fin}(I, J) \in \mathcal{M}$, for some $I, J \in \mathcal{M}$. Write $\mathbb{1}$ for the maximal element $\emptyset \in \mathbb{P}$.

(a) Define, in $\mathcal{M}$, a $\mathbb{P}$-name for the union of the generic: that is, some $\pi \in \mathcal{M}^{\mathbb{P}}$ such that for any $\mathcal{M}$-generic G, $\pi_G = \bigcup G$.

(b) Show that $\mathbb{1} \Vdash \pi$ is a function $I \longrightarrow J$.

(c) Show that for each $f : I \longrightarrow J$ in $\mathcal{M}$, $\mathbb{1} \Vdash \pi \neq \hat{f}$, where $\hat{f}$ denotes the canonical name for f.

You may use whichever definition of $\Vdash$ you prefer — Kunen's, or the inductive definition as we gave it in class — and any theorems about them you require; but note carefully what such facts you use.

—— End of exam —— Slut på provet ——

Errata

There are two typos in the exam MM7048 2026-05-26:

- 1(b) should have $V_{\omega+\omega}$ instead of V_ω.
- 4(c) should assume that I is infinite and $\|J\| \geq 2$.