FINAL EXAM

1. Instructions

Justify your answers. You may use part (a) of a problem to do part (b) even if you have not solved (a), and so on. You may use all results and problems from lectures, exercise sessions and the homeworks, unless you are explicitly asked to solve/prove such a result/exercise. Please carefully state the results you use. You do not need to restate the question in your solution. You may use the character tables of A_4 and S_5 given in Tables 1, 2 below.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Notation throughout this exam: Let F be a field, let V be a finite-dimensional F-vector space and let G be a group. Let $r : G \to \mathsf{GL}(V)$ be a representation of G. Recall that (V, r) is

(a) self-dual if it is isomorphic to its dual;

(b) monomial if it is induced from a one-dimensional representation of a subgroup of G.

Table 1. Character table of S_5

S_5	$1^{(1)}_{(1)}$	$(12)^{(2)}_{(10)}$	$(123)^{(3)}_{(20)}$	$(1234)^{(4)}_{(30)}$	$(12345)^{(5)}_{(24)}$	$(12)(34)^{(2)}_{(15)}$	$(123)(45)^{(6)}_{(20)}$
1	1	1	1	1	1	1	1
sgn	1	-1	1	-1	1	1	-1
χ_4	4	2	1	0	-1	0	-1
χ'_4	4	-2	1	0	-1	0	1
$\wedge^2\chi_4$	6	0	0	0	1	-2	0
χ_5	5	1	-1	-1	0	1	1
χ'_5	5	-1	-1	1	0	1	-1

Table 2. Character table of A_4, where $\zeta^3 = 1$ and $\zeta \neq 1$.

A_4	$1^{(1)}_{(1)}$	$(12)(34)^{(2)}_{(3)}$	$(123)^{(3)}_{(4)}$	$(132)^{(3)}_{(4)}$
1	1	1	1	1
ψ	1	1	ζ	ζ^2
ψ^{-1}	1	1	ζ^2	ζ
χ_3	3	-1	0	0

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2. Problems

There are 5 problems:

Problem 1 (6 points)**.**

(a) (1 point) Define the notion of a quotient representation of (V, r).

(b) (2 points) Give an example of (F, V, G, r) where G is finite and r is not semisimple. Justify your answer.

(c) (2 points) Assume G is finite. Show that every irreducible representation of G is a quotient of the regular representation.

(d) (1 point) Assume V is irreducible and F is algebraically closed. Suppose $A \in \mathsf{GL}(V)$ commutes with $r(g)$ for all $g \in G$. Show that A is scalar (that is, A is a scalar multiple of the identity).

Problem 2 (4 points)**.** *Assume G is finite and $F = \mathbf{C}$.*

(a) (2 points) Show that V is self-dual if and only if its character is real-valued.

(b) (2 points) Show that $x, y \in G$ are conjugate in G if and only if $\chi(x) = \chi(y)$ for all $\chi \in \mathsf{Irr}(G)$.

Problem 3 (7 points)**.** *Assume that G is finite and $F = \mathbf{C}$.*

(a) (2 points) Let N be a normal subgroup of G. Let χ be an irreducible character of N. For all $g \in G \setminus N$, show that

$$(\mathsf{Ind}_N^G \chi)(g) = 0.$$

(b) (2 points) Give a formula for the character of $\mathrm{Sym}^3 V$ in terms of the character of V.

(c) (2 points) Deduce from Table 2 that A_4 has a unique nontrivial, proper normal subgroup N.

(d) (1 point) Use (c) to show that A_4 has no subgroup of order 6.

Problem 4 (5 points)**.** *Assume G is finite and $F = \mathbf{C}$. Suppose G acts on a finite set X.*

(a) (1 point) Define the associated linear representation (V, r) of G.

(b) (1 point) Show that the character χ of V is given by $\chi(g) = |\mathsf{Fix}(g)|$, the number of fixed points of g acting on X.

(c) (3 points) Specialize to $G = S_5$ and $X = \mathsf{Syl}_3(G)$ the set of its 3-Sylow subgroups. Decompose the character of (V, r) into irreducibles.

Problem 5 (8 points)**.** *Let $F = \mathbf{C}$. Let p be a prime and let G be a group of order p^3 which is not abelian.*

(a) (2 points) Show that G has precisely p^2 degree 1 characters.

(b) (3 points) Specialize to $p = 2$. Determine the character table of G.

(c) (2 points) Still assuming $p = 2$, show that every irreducible character of G is monomial.

(d) (1 point) Give an example of two non-isomorphic groups with the same character table. Justify your answer.