FINAL EXAM SOLUTIONS

1. Instructions

Justify your answers. You may use part (a) of a problem to do part (b) even if you have not solved (a), and so on. You may use all results and problems from lectures, exercise sessions and the homeworks, unless you are explicitly asked to solve/prove such a result/exercise. Please carefully state the results you use. You do not need to restate the question in your solution. You may use the character tables of A_4 and S_5 given in Tables 1, 2 below.

Grading: This exam is worth 30 points. If you completed homework assignments, your homework bonus (out of 3 points) will be added to your score. You need a score of 12.5/30 or higher to pass this exam. More precisely, the following scale will be used:

A: $[26.5, 30]$, B: $[23, 26.5)$, C: $[19.5, 23)$, D: $[16, 19.5)$, E: $[12.5, 16)$, F: $[0, 12.5)$.

Notation throughout this exam: Let F be a field, let V be a finite-dimensional F-vector space and let G be a group. Let $r : G \to \mathsf{GL}(V)$ be a representation of G. Recall that (V, r) is

(a) self-dual if it is isomorphic to its dual;

(b) monomial if it is induced from a one-dimensional representation of a subgroup of G.

Table 1. Character table of S_5

S_5	$1^{(1)}_{(1)}$	$(12)^{(2)}_{(10)}$	$(123)^{(3)}_{(20)}$	$(1234)^{(4)}_{(30)}$	$(12345)^{(5)}_{(24)}$	$(12)(34)^{(2)}_{(15)}$	$(123)(45)^{(6)}_{(20)}$
$\mathbf{1}$	1	1	1	1	1	1	1
sgn	1	-1	1	-1	1	1	-1
χ_4	4	2	1	0	-1	0	-1
χ'_4	4	-2	1	0	-1	0	1
$\wedge^2\chi_4$	6	0	0	0	1	-2	0
χ_5	5	1	-1	-1	0	1	1
χ'_5	5	-1	-1	1	0	1	-1

Table 2. Character table of A_4, where $\zeta^3 = 1$ and $\zeta \neq 1$.

A_4	$1^{(1)}_{(1)}$	$(12)(34)^{(2)}_{(3)}$	$(123)^{(3)}_{(4)}$	$(132)^{(3)}_{(4)}$
$\mathbf{1}$	1	1	1	1
ψ	1	1	ζ	ζ^2
ψ^{-1}	1	1	ζ^2	ζ
χ_3	3	-1	0	0

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2. Problems

There are 5 problems:

Problem 1 (6 points)**.**

(a) (1 point) Define the notion of a quotient representation of (V, r).
(b) (2 points) Give an example of (F, V, G, r) where G is finite and r is not semisimple. Justify your answer.
(c) (2 points) Assume G is finite. Show that every irreducible representation of G is a quotient of the regular representation.
(d) (1 point) Assume V is irreducible and F is algebraically closed. Suppose $A \in \mathsf{GL}(V)$ commutes with $r(g)$ for all $g \in G$. Show that A is scalar (that is, A is a scalar multiple of the identity).

Solution: (a): A quotient of (V, r) is defined for every G-stable subspace $W \subset V$ by

$$V/W \xrightarrow{g} V/W$$

$$v + W \longmapsto gv + W$$

for every $v + W \in V/W$. This is well defined since W is G-stable: For all $v \in V$ and $w \in W$, one has $g(v + w) - gv = gw \in W$.

(b): Let p be a prime, $G = \mathbf{Z}/p$ cyclic of order p, $F = \mathbf{F}_p$ (any field of characteristic p works equally well) and r uniquely determined by

$$r(1) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This is well-defined since for all $n \in \mathbf{Z}$

$$r(1) := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

In particular, $r(p) = I$ since F has characteristic p. The characteristic polynomial of an upper-triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ is $\prod_{i=1}^n (x - \lambda_i)$. So that of $r(1)$ is $(x-1)^2$. Hence 1 is the unique eigenvalue of $r(1)$, even over an algebraically closed field containing $\mathbf{F}_p$. Hence the unique G-stable subspace of (V, r) is the 1-eigenspace $V_1 = \mathrm{span}(e_1)$ of dimension $\dim V_1 = 1$. Hence $r(1)$ is not diagonalizable and r is not semisimple.

(c): For a nonzero G-representation V and nonzero $v \in V$, define the evaluation φ at v:

$$F[G] \xrightarrow{\varphi} V$$

$$\alpha \longmapsto \alpha v.$$

The action axiom $(g\alpha)v = g(\alpha v)$ states that φ is G-equivariant. Its image is nonzero since $\varphi(1) = v$ is so. If V is irreducible, this implies the image of φ is all of V. So φ is surjective and V is the quotient of $F[G]$ by $\ker \varphi$.

(d) Since A commutes with $r(G)$ it is a G-equivariant map: $A \in \mathrm{End}_G(V)$. Since F is algebraically closed and V is irreducible, Schur's Lemma implies that $\mathrm{End}_G(V) = \{\lambda I \mid \lambda \in F\}$. □

Problem 2 (4 points)**.** *Assume G is finite and $F = \mathbf{C}$.*

(a) (2 points) Show that V is self-dual if and only if its character is real-valued.
(b) (2 points) Show that $x, y \in G$ are conjugate in G if and only if $\chi(x) = \chi(y)$ for all $\chi \in \mathsf{Irr}(G)$.

Solution: (a) Let χ be the character of r. In a basis B of V, the dual representation $r^\vee$ satisfies $r^\vee(g) = {}^t r(g)^{-1}$. Let $n = \dim V$. Then r is self-dual if and only if there exists $A \in \mathsf{GL}(n, \mathbf{C})$ such that $Ar(g)A^{-1} = r^\vee(g)$ for all $g \in G$. A matrix and its transpose have the same trace. Since $\chi(g^{-1}) = \overline{\chi(g)}$ the character of $r^\vee$ is $\overline{\chi}$, the complex-conjugate of χ. Since two representations of a finite group over $\mathbf{C}$ are isomorphic if and only if they have the same character, r is self-dual if and only if $\chi = \overline{\chi}$ is real-valued. (b) Since $\mathrm{tr}AB = \mathrm{tr}BA$, every character (irreducible or not) is a class function.

The interesting direction is to assume that $\chi(x) = \chi(y)$ for all $\chi \in \mathsf{Irr}(G)$. Recall that for all $x, y \in G$,

$$[x, y] := \sum_{\chi \in \mathsf{Irr}(G)} \chi(x)\overline{\chi(y)}.$$

The second orthogonality relation states that,

$$[x,y] = \begin{cases} |\operatorname{Cent}_G(x)| & \text{if } x,y \text{ are conjugate in } G \\ 0 & \text{otherwise.} \end{cases}$$

The assumption gives $[x,y] = [x,x] = |\operatorname{Cent}_G(x)|$. Every centralizer $\operatorname{Cent}_G(x)$ has order ≥ 1 since it contains 1 (and x and possibly $x = 1$). Since $[x,y] \geq 1$, the elements x, y are conjugate in G. □

Problem 3 (7 points)**.** *Assume G is finite and $F = \mathbf{C}$.*

(a) (2 points) Let N be a normal subgroup of G. Let χ be an irreducible character of N. For all $g \in G \setminus N$, show that

$$(\mathsf{Ind}_N^G \chi)(g) = 0.$$

(b) (2 points) Give a formula for the character of $\operatorname{Sym}^3 V$ in terms of the character of V.
(c) (2 points) Deduce from Table 2 that A_4 has a unique nontrivial, proper normal subgroup N.
(d) (1 point) Use (c) to show that A_4 has no subgroup of order 6.

Solution: (a) There are two equivalent formulas for the induced character: Recall that χ^0 is the extension of the function χ to G by zero, meaning that $\chi^0(g) = \chi(g)$ if $g \in N$ and $\chi^0(g) = 0$ otherwise. If $g_1, \ldots, g_m$ is a set of representatives of left cosets of N in G, then for all $g \in G$,

$$\mathsf{Ind}_N^G \chi(g) = \sum_{i=1}^{m} \chi^0(g_i^{-1} g g_i).$$

Equivalently, without choosing representatives,

$$\mathsf{Ind}_N^G \chi(g) = \frac{1}{|H|} \sum_{h \in G} \chi^0(h^{-1} g h).$$

If N is normal in G and $g \notin N$, then also no conjugate of g lies in N. So every term in both formulas is zero when $g \notin N$ and N is normal in G.

(b): Let χ (resp. ψ be be the character of V (resp. $\wedge^3 V$). Let $g \in G$. Since $F = \mathbf{C}$ and G is finite, g is diagonalizable. So there exists a basis of g-eigenvectors $e_1, \ldots, e_n$ of V and $\lambda_1, \ldots \lambda_n \in \mathbf{C}$ such that $g e_i = \lambda_i e_i$. A basis of $\operatorname{Sym}^3 V$ is given by the images of

$$\{e_i \otimes e_j \otimes e_k | 1 \leq i \leq j \leq k \leq n\}$$

in $\operatorname{Sym}^3 V$ under the projection $V^{\otimes 3} \twoheadrightarrow \operatorname{Sym}^3 V$ defining $\operatorname{Sym}^3 V$. Hence

$$\psi(g) = \sum_{1 \leq i \leq j \leq k \leq n} \lambda_i \lambda_j \lambda_k.$$

Since $\chi(g) = \sum_{i=1}^n \lambda_i$, the value at the jth power is $\chi(g^j) = \sum_{i=1}^n \lambda_i^j$ for all $j \geq 1$. Hence

$$\chi(g)^3 = (\sum \lambda_i)^3 = \sum_i \lambda_i^3 + 3 \sum_{i<j} \lambda_i \lambda_j + 6 \sum_{i<j<k} \lambda_i \lambda_j \lambda_k.$$

So

$$\chi(g)^3 + 3\chi(g)\chi(g^2) + 2\chi(g^3) = 6\psi(g).$$

Dividing by 6:

$$\psi(g) = \frac{\chi(g)^3 + 3\chi(g)\chi(g^2) + 2\chi(g^3)}{6}.$$

(d) Every normal subgroup of G is an intersection of kernels $\ker \chi = \{g \in G | \chi(g) = \chi(1)\}$ of irreducible characters χ. Inspecting 2, the three irreducible characters with nontrivial kernel are precisely the degree 1 characters. Their kernels are all equal to V_4, the disjoint union of the identity and the $(2,2)$ class represented in the table by $(12)(34)$. Hence the only proper, nontrivial intersection of kernels is V_4.

(d) Assume for a contradiction that H is an order 6 subgroup of A_4. Since H has index 2, it is normal. This contradicts (c), since V_4 has order 4. □

Proof.

Remark 1. 'No order 6 subgroup in A_4' is the the smallest counterexample to the converse of Lagrange's Theorem. A standard Sylow-theoretic proof without characters that A_4 has no order 6 subgroup is: If A_4 has such a subgroup H, then H has a normal, hence characteristic 3-Sylow subgroup. Since H is normal in A_4 and since

$$A \text{ characteristic in } B \text{ normal in } C \implies A \text{ normal in } C$$

this would imply that A_4 has a normal 3-Sylow subgroup, contradicting its eight 3-cycles making up its 4 Sylow 3-subgroups.

□

Problem 4 (5 points)**.** *Assume G is finite and $F = \mathbf{C}$. Suppose G acts on a finite set X.*

(a) (1 point) Define the associated linear representation (V, r) of G.
(b) (1 point) Show that the character χ of V is given by $\chi(g) = |\mathsf{Fix}(g)|$, the number of fixed points of g acting on X.
(c) (3 points) Specialize to $G = S_5$ and $X = \mathsf{Syl}_3(G)$ the set of its 3-Sylow subgroups. Decompose the character of (V, r) into irreducibles.

Solution: (a): Let $\mathbf{C}^X$ be the $\mathbf{C}$-vector space with basis X. So elements of $\mathbf{C}^X$ are linear combinations $\sum_{xinX} a_x x$ with coefficients $a_x \in \mathbf{C}$. The associated linear representation $G \to \mathsf{GL}(\mathbf{C}^X)$ is given by extending the action of G on X to $\mathbf{C}^X$ by linearity. So

$$g(\sum_{x\in X} a_x x) = \sum_{x\in X} a_x gx.$$

(b) The value $\chi(g)$ is the trace of the linear action of g on $\mathbf{C}^X$ defined in (a). Relative to the basis X, the element g acts by a permutation matrix (that is, every basis element is mapped to a basis element). Hence its trace is the number of fixed points.

(c) Let $P = \langle(123)\rangle$. Since $|S_5| = 5!$ is not divisible by 3^2, the subgroup P is a 3-Sylow subgroup of S_5. Every distinct pair of 3-Sylows intersects trivially, and every 3-Sylow consists of two 3-cycles and the identity. Hence

$$|\mathsf{Syl}_3(S_5)| = 5!/(3.2) = 10.$$

Sylow check: $10 = 1 \pmod 3$.

Let $N = N_{S_5}(P)$. Then $|N| = 120/10 = 12$.

Since (45) is disjoint from (123), the two commute. So $(45) \in N$. Within S_3, the subgroup $P = A_3$ is normal in S_3. Hence $S_3 = \mathsf{Fix}(\{4, 5\}) \subset N$. Thus

$$N = \langle(123), (12), (45)\rangle = S_3 \sqcup (45)N.$$

In particular, N has $1, 4, 2, 3, 2$ elements of cycle type trivial,2,3,$(2, 2)$, $(2, 3)$ and order $1, 2, 3, 2, 6$ respectively.

Let χ be the permutation character we want to decompose into irreducibles. Then $\chi(1) = |X| = 10$ and $\chi(g) = 0$ if g is not of one of the cycle types present in N.

For the cycle types in N, the Burnside counting lemma gives

$$\chi(g) = \frac{|\operatorname{Class}(g) \cap N||\mathsf{Syl}_3(S_5)|}{|\operatorname{Class}(g)|}.$$

Aided by Table 1, one finds

$$\chi((12)) = \frac{4.10}{10} = 4, \quad \chi((123)) = \frac{2.10}{20} = 1, \quad \chi((12)(34)) = \frac{3.10}{15} = 2, \text{ and } \chi((123)(45)) = \frac{2.10}{20} = 1.$$

Hence the extended Table 3.

Since a group G acts transitively on $\mathsf{Syl}_p(G)$, Homework 1 implies $(\langle\chi, \mathbf{1}) = 1$. This is also checked using the definition and the table:

$$(chi, \mathbf{1}) = (10 + 4.10 + 1.20 + 2.15 + 1.20)/120 = 1.$$

Since

$$(\chi, \chi) = (10^2 + 10.4^2 + 20.1^2 + 15.2^2 + 20.1^1)/120 = 3,$$

Table 3 shows that $\chi - \mathbf{1}$ is the sum of an irreducible degree 5 character and an irreducible degree 4 character. Visibly the pairings of χ with χ_5 and χ_4 are larger than with χ_5' and χ_4'. Hence

$$\chi = \mathbf{1} + \chi_4 + \chi_5.$$

Check (good for finding errors but logically unnecessary):

$$(\chi, \chi_4) = (10.4 + 4.2.10 + 1.1.20 + 1.(-1).20)/120 = 1$$

TABLE 3. Character table of S_5 extended by the permutation character χ

S_5	$1^{(1)}_{(1)}$	$(12)^{(2)}_{(10)}$	$(123)^{(3)}_{(20)}$	$(1234)^{(4)}_{(30)}$	$(12345)^{(5)}_{(24)}$	$(12)(34)^{(2)}_{(15)}$	$(123)(45)^{(6)}_{(20)}$
1	1	1	1	1	1	1	1
sgn	1	-1	1	-1	1	1	-1
$\chi_4 = \chi_{4,\mathrm{Std}}$	4	2	1	0	-1	0	-1
χ_4'	4	-2	1	0	-1	0	1
$\wedge^2\chi_4$	6	0	0	0	1	-2	0
χ_5	5	1	-1	-1	0	1	1
χ_5'	5	-1	-1	1	0	1	-1
χ	10	4	1	0	0	2	1

$$(\chi, \chi_5) = (10.5 + 4.1.10 + 1.(-1).20 + 2.1.15 + 1.1.20)/120 = 1.$$

□

Problem 5 (8 points). *Let $F = \mathbf{C}$. Let p be a prime and let G be a group of order p^3 which is not abelian.*

(a) (2 points) Show that G has precisely p^2 degree 1 characters.
(b) (3 points) Specialize to $p = 2$. Determine the character table of G.
(c) (2 points) Still assuming $p = 2$, show that every irreducible character of G is monomial.
(d) (1 point) Give an example of two non-isomorphic groups with the same character table. Justify your answer.

Solution: (a): For a group H, we know from abstract algebra that if H/Z_H is cyclic, then H is abelian. Since G is not abelian, G/Z_G is not cyclic. Since every group of order p is cyclic and every group of order p^2 is either cyclic or isomorphic to $\mathbf{Z}/p \times \mathbf{Z}/p$, the center Z_G has order p and $G/Z_G \cong \mathbf{Z}/p \times \mathbf{Z}/p$. Since the commutator subgroup $G' \subset G$ is contained in every normal subgroup N such that G/N is abelian, $G' \subset Z_G$. Since G is not abelian $G' \neq 1$. Hence $G' = Z_G$. Since G/G' is the number of degree 1 characters over $\mathbf{C}$, G has p^2 such characters.

(b): By (a), there are precisely 4 degree 1 characters, say $\mathbf{1}, \psi_1, \psi_2, \psi_3$. They are all trivial on the order 2 center Z_G since $Z_G = G'$ by (a). Since every finite subgroup of the multiplicative group of a field is cyclic, the ψ_i are not faithful characters of $G/G' = \mathbf{Z}/2 \times \mathbf{Z}/2$. Therefore each ψ_i acts trivially on one nontrivial element of $\mathbf{Z}/2 \times \mathbf{Z}/2$ and non-trivially on the other two.

Since G is not abelian, it has a character χ of degree $\chi(1) > 1$. Since the sum of the squares of the degrees of the irreducible characters is the order of the group, $\chi(1) = 2$ and χ is the unique character of degree > 1.

Write $Z_G = \{1, z\}$. Since the number of irreducible representations equals the number of conjugacy classes, there are three non-central conjugacy classes in G. Let a, b, c be representatives of these classes. Let ψ_1, ψ_2, ψ_3 be the three nontrivial degree 1 characters. This gives the partial table:

There are many ways to fill in the last χ-row. One way is to apply the second orthogonality relation to each pair of columns containing the first one. Since $Z_G = \{1, z\}$, the centralizer of z has order 8, while a, b, c all have centralizers of order 4 since they commute with z but are not central.

Hence the completed table:

(c) It suffices to show that χ is monomial. Since $\chi(1) = 2$, if it is monomial, it is induced from an index 2 subgroup. Let H be such a subgroup. Explicitly, $H = \langle z, x \rangle$ for $x = a, b$ or c (the three options give three order 4 subgroups). There is a degree 1 character ξ of H such that $\xi(z) = -1$ (in fact there are two such) since the characters of H that are trivial on z factor through H/Z_G and H/Z_G only has two characters. Then $\mathsf{Ind}_H^G \xi(z) = [G : H].\xi(z) = -2$. Combined with the table, this shows that $\mathsf{Ind}_H^G \xi$ is not a sum of two (possibly equal) degree 1 characters. Hence $\mathsf{Ind}_H^G \xi = \chi$.

TABLE 4. Partial table for G of order 8

G	$1^{(1)}_{(1)}$	$z^{(2)}_{(1)}$	a	b	c
$\mathbf{1}$	1	1	1	1	1
ψ_1	1	1	1	-1	-1
ψ_2	1	1	-1	1	-1
ψ_3	1	1	-1	-1	1
χ	2				

TABLE 5. Completed table for G of order 8

G	$1^{(1)}_{(1)}$	$z^{(2)}_{(1)}$	a	b	c
$\mathbf{1}$	1	1	1	1	1
ψ_1	1	1	1	-1	-1
ψ_2	1	1	-1	1	-1
ψ_3	1	1	-1	-1	1
χ	2	-2	0	0	0

(d) The dihedral and quaternion groups of order 8 are not abelian and non-isomorphic. For example, the dihedral group has 5 elements of order 2 while $z = -1$ is the unique order 2 element of the quaternion group of order 8. By (b), they have the same character table. □