

Kursrapport MM7050 Representationsteori för ändliga grupper VT26

Antal respondenter: 1
Antal svar: 1
Svarsfrekvens: 100,00 %

. Beskrivning av kursupplägget.

Beskrivning av kursupplägget.

The course is an introduction to the representation theory of finite groups, focusing on characters over the complex numbers. It starts with some general notions concerning representations of a group G on a vector space V over a field F , which apply much more broadly than the setting adopted later of finite G , $F=\mathbb{C}$ =complex numbers and V finite-dimensional. These notions include representations, linear group actions, sub, quotient, sub-quotient and dual representations, direct sums and semisimplicity. This initial stage also illustrates how representations of cyclic groups and whether they are semisimple corresponds to the linear algebra of linear transformation and whether it is diagonalizable. This is used to give examples where theorems announced in the introduction and proven later about the case G finite and $F=\mathbb{C}$ fail if either assumption is removed. A key result proved at this stage, without assuming G is finite, is Schur's Lemma. It is a deeper connection between linear algebra and representation theory. Schur's Lemma allows to deduce very early on that a finite-dimensional irreducible representation of an abelian group (not necessarily finite) over an algebraically closed field is one-dimensional.

Restricting to finite groups G , the first key result proved is Maschke's Theorem, that representations of G are semisimple when the characteristic of the field does not divide the order of G . The next stage of the course introduces algebras and modules and the key example of the group algebra $F[G]$. Tensor products of modules and representations are also introduced. The course studies some Wedderburn theory of finite-dimensional semisimple algebras and applies it to $F[G]$ under the assumptions of Maschke's Theorem. The key is Wedderburn's Theorem that, under these assumptions, $F[G]$ is a direct product of matrix algebras over F -division algebras.

Part 3 of the course applies the ring-theoretic structure of $F[G]$ to the characters of G . Results are often proved over $F=\mathbb{C}$ or algebraically closed of characteristic 0, but mentioned to be true in the more general setting where F is algebraically closed of characteristic not dividing $|G|$. In particular, this includes deducing the orthogonality relations, decomposing the regular representation and its character and showing that, over F algebraically closed of characteristic 0, the character determines the isomorphism class of the representation.

Later topics include induced representations and characters, illustrating how permutation representations and characters are a special case. Techniques including induced representations, tensor products (as well as alternating and symmetric powers) and the orthogonality relations are used to compute examples of character tables, such as the symmetric groups S_3 , S_4 , S_5 , alternating groups A_4 , A_5 and groups of order p^3 . The latter illustrates that non-isomorphic groups may have the same character table. Some comments are given about the additional structure present for finite groups of Lie type, focusing on the examples of $SL(2, p)$ and $GL(2, p)$.

As an extra topic for this occurrence of the course, the last couple of lectures introduced the McKay correspondence via computational examples, illustrating (a) further examples of character tables, e.g. for $SL(2, 3)$, (b) how a character table for G can sometimes be constructed when G itself is not fully known, e.g. the double cover of S_4 in $SL(2, \mathbb{C})$, and how the representation theory of finite groups occupies a central position in mathematics, with connections to numerous other fields, many at the cutting-edge of research today.

. Kursens fördelar, beakta studenternas uppfattning i kursutvärderingar.

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Representation theory is one of the deepest areas in mathematics. It has found groundbreaking applications to virtually every branch of math. This course offers an opportunity to introduce students to this deep, beautiful subject via the special case of finite groups. An advantage of mainly focusing on finite groups is that it requires minimal prerequisites. For the downside of restricting to finite groups, see below in "Course cons".

Building on the applications to other subjects, representation theory has been at the heart of the leading unificatory programs in mathematics over the past half-century, notably the Langlands Program and Grothendieck's Motives Program. The role of representation theory in our collective mathematical understanding continues to rise day by day. Related to this, the course is probably the Master's course which does the best job at giving students a glimpse towards research.

At a more basic level, the connections to other subjects means that the course is an excellent application of the topics learned in Linear Algebra and Abstract Algebra courses.

The course offers excellent preparation and motivation for students to study other kinds of representation theory, such as representations of algebraic groups, Lie groups, Lie algebras etc.

. Kursens nackdelar, beakta studenternas uppfattning i kursutvärderingar.

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Students' background on groups, rings, fields and linear algebra illustrated that these precursor courses need to be more advanced. In particular, students' background in group theory was overall very low, suggesting that the level has decreased while I have not been teaching that course. This is confirmed by the very easy Abstract Algebra exams which were shared with me as course head for that course. Those exams covered either nothing or very little about Sylow theory and group actions, and this was reflected in students' lack of preparation for "Representation theory of finite groups".

It is unfortunate that the course cannot fully assume that students are familiar with algebras, modules and tensor products. It takes considerable time to introduce these concepts.

While the focus on complex representations of finite groups has its limited advantages (mentioned in "Course benefits"), it has more significant disadvantages: It gives a misleading idea of a world where groups are finite, representations are semisimple and finite-dimensional and everything can be done with characters. In fact, most groups are very wild and infinite, most representations are not semisimple and many are naturally infinite-dimensional. As mentioned above, I try to rectify this by using linear algebra to briefly illustrate nature's complexity via some elementary examples.

Even more importantly, it is difficult to illustrate several rich features of representation theory with finite groups. For example, the theory of weights, "highest weight theorems" and "families of representations" (e.g. variations of the highest weight) arise more directly for reductive groups and their Lie algebras. The latter representation theory is also more directly and ubiquitously tied to algebraic geometry and the leading research programs mentioned above.

Overall, the level of the students was considerably lower than when I taught this course in 2018 and 2020. This is in-line with a general drop in the level of Master's students over the past few years. One may question whether this may be related to

(a) The fact that it has unfortunately become harder for students from other countries to study at Stockholm University, and consequently to take the course. Talented students who may be interested in studying at Stockholm University and taking this course face greater uncertainty about the whole process of moving to Stockholm, registering and so on all the way to taking the course.

(b) There seems to be less support for the more talented students who are more interested in continuing in math, regardless of where they are from.

. Slutsatser samt förslag till förbättringar.

Slutsatser samt förslag till förbättringar.

1. Resume offering the Master's course "Reductive algebraic groups" ever other year: It nicely complements the course under review and addresses many of the comments under "Course cons".
2. Make the earlier algebra courses, such as "Abstract Algebra" and "Linear Algebra" more advanced.
3. The course "Advanced Algebra" should be added as a prerequisite for this course, and it should contain more material that's useful for this course as well as for Galois theory. One semester is not enough for giving students a reasonable foundation in "Abstract Algebra". Many universities offer at least a whole year on this topic.
4. Make it easier for students from other countries to take the course, and offer more support to all talented students who want to continue in math.