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Suggested solutions

Exam: Introduction to Finance Mathematics (MT5009), 2026-05-25

Problem 1

(A)
$$10 \cdot e^{-0.05\cdot 1} + 10 \cdot e^{-0.05\cdot 2} + (10 + 200) \cdot e^{-0.05\cdot 3} \approx 199.31.$$

(B) Immediately after the first coupon is paid the bond has two years remaining to maturity, and its value is

$$10 \cdot e^{-0.05\cdot 1} + (10 + 200) \cdot e^{-0.05\cdot 2} \approx 199.53.$$

Problem 2

(A) First we find

$$S^u = 5 \cdot 1.2 = 6, \qquad S^d = 5 \cdot 0.9 = 4.5.$$

Now we use the payoff function to find

$$f(S^u) = \max(6^2 - 4 \cdot 6, 0) = 12, \qquad f(S^d) = \max(4.5^2 - 4 \cdot 4.5, 0) = 2.25.$$

We find risk-neutral probabilities

$$p^* = \frac{R - D}{U - D} = \frac{0.05 - (-0.1)}{0.2 - (-0.1)} = \frac{0.15}{0.3} = 0.5, \qquad 1 - p^* = 0.5.$$

The value of derivative in case it is European is therefore

$$\begin{aligned} H_E(0) &= \frac{1}{1+R}\left[p^* f(S^u) + (1-p^*) f(S^d)\right] \\ &= \frac{1}{1.05}\left[0.5 \cdot 12 + 0.5 \cdot 2.25\right] = 6.79. \end{aligned}$$

(B) Let us first find the value of derivative in case it is American:
First note that immediate exercise would yield the value:

$$f(S_0) = \max(5^2 - 4 \cdot 5, 0) = 5.$$

Since
$$H_E(0) = 6.79 > f(S_0) = 5,$$

we conclude that it is optimal not to exercise the American derivative early. Hence
$$H_A(0) = H_E(0) = 6.79.$$

Let us now find the replicating portfolio:

The replicating portfolio satisfies by definition

$$x(1)S^u + y(1)(1+R) = f(S^u)$$

$$x(1)S^d + y(1)(1+R) = f(S^d),$$

where $x(1)$ is the number of shares and $y(1)$ is the amount of money in the risk-free asset in the replicating portfolio.

By plugging in numbers for $S^u, f(S^u), S^d, f(S^d), R$ (see above) we see that the replicating portfolio should satisfy

$$x(1) \cdot 6 + y(1) \cdot 1.05 = 12$$

$$x(1) \cdot 4.5 + y(1) \cdot 1.05 = 2.25.$$

Solving this equation system gives the replicating portfolio

$$x(1) = 6.5 \quad \text{and} \quad y(1) = -25.71.$$

Problem 3

The mathematical problem corresponding to finding the market portfolio is

$$\max_w \frac{wm^T - R}{\sqrt{wCw^T}}$$

(where $w = (w_1, w_2)$) under the condition $w_1 = 1 - w_2$ (see Capinski & Zastawniak circa p. 82 for an explanation/motivation). Set $s = w_1$ so that $w_2 = 1 - s$.

Let us now solve the problem: Using the above and the information in the problem formulation we have

$$wm^T = s\mu + (1-s)\mu = \mu$$

and

$$wCw^T = s^2\sigma_1^2 + (1-s)^2\sigma_2^2.$$

Using also that $R = 0$, we find that our problem can be written as

$$\max_s \frac{\mu}{\sqrt{s^2\sigma_1^2 + (1-s)^2\sigma_2^2}}.$$

which (since $\mu > 0$) is equivalent to

$$\min_s D(s)$$

where

$$D(s) = s^2\sigma_1^2 + (1-s)^2\sigma_2^2.$$

(i.e. our problem is in this special case equivalent to finding the minimum variance portfolio). We find

$$D'(s) = 2s\sigma_1^2 - 2(1-s)\sigma_2^2.$$

Hence, the FOC $D'(s) = 0$ (which clearly gives the minimum) yields

$$D'(s) = 2s\sigma_1^2 - 2(1-s)\sigma_2^2 = 0.$$

Solving for s in the FOC yields:

$$2s\sigma_1^2 - 2(1-s)\sigma_2^2 = 0$$

$$2s\sigma_1^2 - 2\sigma_2^2 + 2s\sigma_2^2 = 0$$

$$2s(\sigma_1^2 + \sigma_2^2) = 2\sigma_2^2$$

$$s^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

We conclude that the market portfolio weights are

$$w_M = (s^*, 1-s^*) = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right)$$

Note that the portfolio puts more weight on the less risky asset (i.e., whichever one of the assets has the lower variance). This should be in line with intuition, since both assets have the same expected return.

Problem 4

(A) Using the parameters in the problem formulation we find

$$p^* = \frac{R-D}{U-D} = 0.75, \quad 1-p^* = 0.25,$$

and the share values and call payoffs at $t = 2$:

$$S^{uu} = 100(1.1)^2 = 121, \quad S^{ud} = 100 \cdot 1.1 \cdot 0.9 = 99, \quad S^{dd} = 100(0.9)^2 = 81,$$

$$C^{uu} = \max\{S^{uu} - X; 0\} = 11, \quad C^{ud} = \max\{S^{ud} - X; 0\} = 0, \quad C^{dd} = \max\{S^{dd} - X; 0\} = 0.$$

Since only the value C^{uu} exceeds 0, we have (using the risk-neutral valuation formula) that

$$C_E(0) = \frac{1}{(1+R)^2}(p^*)^2 \cdot C^{uu} = 5.61.$$

(B) At $t = 4$ the largest possible share price and the associated derivative payoff are

$$S^{uuuu} = 100(1+0.1)^4 = 146.41, \quad C^{uuuu} = \max\{S^{uuuu} - X\} = \max\{146.41 - 140; 0\} = 6.41.$$

The second largest possible share price and the associated derivative payoff are

$$S^{uuud} = 100(1+0.1)^3(1-0.1) = 119.79, \quad C^{uuud} = \max\{S^{uuud} - X; 0\} = \max\{119.79 - 140; 0\} = 0.$$

Since the remaining possible share prices are all lower than S^{uuud} it holds that also they are lower than the strike $X = 140$, and we hence conclude that all derivative payoffs except C^{uuuu} are 0. Thus, using the risk-neutral valuation formula, we find

$$C_E(0) = \frac{1}{(1+R)^4}(p^*)^4 \cdot C^{uuuu} = 1.67.$$

Problem 5

(A) The forward price is

$$F(0,T) = \frac{S(0)}{B(0,T)} = S(0)e^{rT},$$

which can be shown as in Capinski & Zastawniak, around pages 93–94 (as well as page 44).

(B) The payoff of the forward contract at the time of maturity T is

$$S(T) - F(0,T) = S(T) - X.$$

(since we have $F(0,T) = X$ in the present case, according to the problem formulation). The payoff of the European call option is

$$\max\bigl(S(T) - X, 0\bigr).$$

Hence

$$\max\bigl(S(T) - X, 0\bigr) \geq S(T) - X,$$

meaning that the call option payoff is always at least as large as the forward payoff. By the no-arbitrage principle, it must therefore be the case that the call option has a value that is greater than, or equal to, the value of the forward contract (for any given time $t \in [0,T]$).