Topology, Fall 2025

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Homework exercise #5
Due: Thursday Nov 13, 2025

Instructions: We strongly prefer that you typeset your answers with Latex or similar. Handwritten assignments will be graded only if easy to read.

You may work together on the assignment, but write your own solution and do not copy. You also are welcome to consult the instructors, after you made a decent effort on a problem. Tell us what you have tried, and we will be happy to provide some direction if needed.

Please, please, do not seek inappropriate help on the internet or from ChatGPT and the like.

Additional notes: you are supposed to solve this problem with methods learned in this course. So, no using homology and no quoting of fancy theorems that are not part of the course. You may quote results from the relevant chapters of the textbook even if they were not mentioned in class.
Also, later parts of the problem may depend on earlier parts. You may use the results claimed in earlier parts even if you did not do them.

Recall that $\mathbb{R}P^n$ is the quotient space of S^n by the antipodal action of $\mathbb{Z}/2$. Let $q_n \colon S^n \to \mathbb{R}P^n$ be the quotient map.

1. Suppose that $m, n > 1$. Let $f \colon \mathbb{R}P^m \to \mathbb{R}P^n$ be a map. Prove that there exists a map $\tilde{f} \colon S^m \to S^n$ that satisfies $q_n \circ \tilde{f} = f \circ q_m$. How many such maps are there? (10 points)

   Hint: Theorem 11.18 or one of its corollaries is likely to be useful here.

2. Let $\tilde{f}$ be as in the previous part. Show that either $\tilde{f}(-x) = \tilde{f}(x)$ for all $x \in S^m$ or $\tilde{f}(-x) = -\tilde{f}(x)$ for all $x \in S^m$. We call a map $\tilde{f}$ that satisfies one of these properties *even* or *odd* respectively. How does the evenness/oddness of $\tilde{f}$ relate to the homomorphism $f_* \colon \pi_1(\mathbb{R}P^m) \to \pi_1(\mathbb{R}P^n)$? (10 points)

3. Now suppose that we have a map $g \colon S^1 \to S^1$. Then g induces a homomorphism $g_* \colon \mathbb{Z} \to \mathbb{Z}$ between fundamental groups. Every such homomorphism has the form $g_*(n) = dn$ for some integer d. We call d the *degree* of g.

   Prove that if g is even then the degree of g is even, and if g is odd then the degree of g is odd. (10 points)

   Note: This is problems 11-12, 11-13 in the book. There is a hint in the book.

4. Use part (3) to prove that there does not exist an odd map $f \colon S^2 \to S^1$. (10 points)

5. Use part (4) to prove the following: suppose we have a map $f \colon S^2 \to \mathbb{R}^2$. Then there exists a point $x \in S^2$ such that $f(x) = f(-x)$.

   A popular interpretation of this result: there always exist two antipodal points on earth that have the same temperature and air pressure. (10 points)