Algorithms and Data Structures

Part 1: Fundamentals

Department of Mathematics
Stockholm University

Part 1 focuses on the following topics.

1-1 What is an algorithm?

1-2 Correctness of algorithms

1-3 Runtime of algorithms & space complexity

The latter Parts **1-1**, **1-2**, **1-3** will, in particular, be examined on a sorting algorithm `Insertion_Sort`. Further examples will be provided. We then continue with

1-4 Elementary Data Structures

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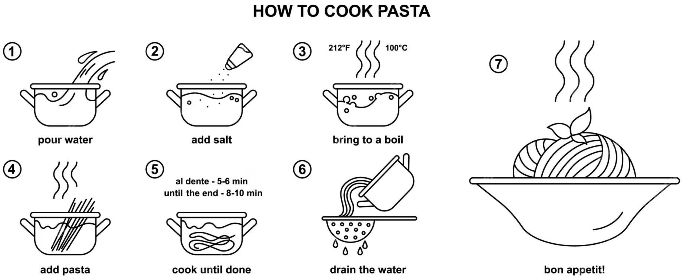

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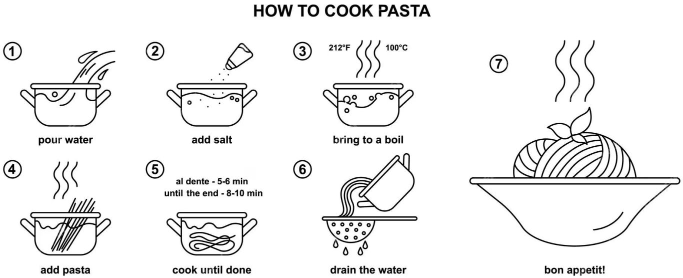

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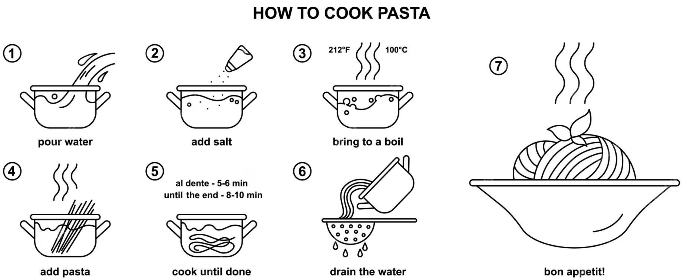

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A human may know how to "boil-up" water by using a cooking plate (...or open fire ...?) but does a robot know this?

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1 Align cards to a pack-of-cards

2 Shuffle cards 3 times

3 Spread cards

4 `IF` (*cards are ordered*) `THEN`
goto step 5
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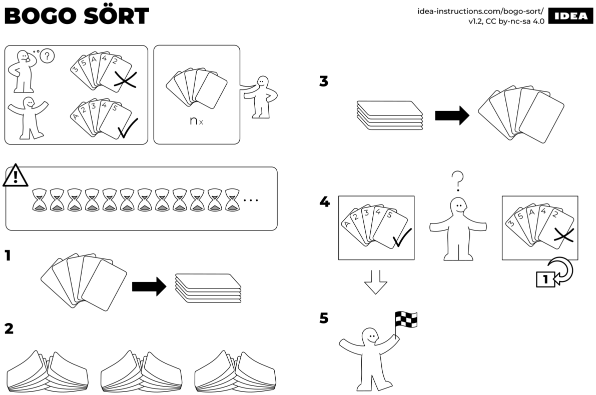
5 `RETURN` Sorted Cart Deck



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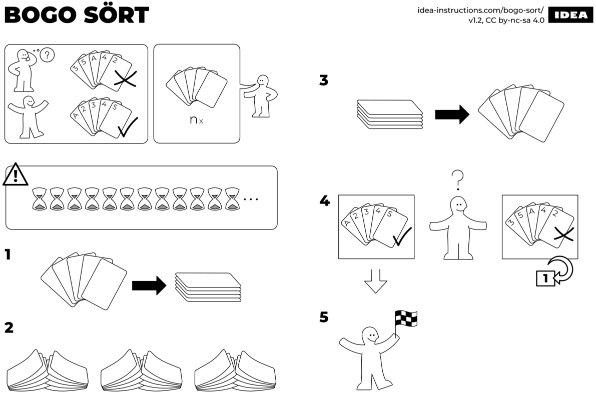

Question: unambiguous (is "order" well-defined)? does it terminate (runtime)?...

---

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Part 1-1: What is an algorithm?

(780–850) Persian mathematician, astronomer, geographer, . . .

The word 'algorithm' has its roots in the name of Persian mathematician Muhammad ibn Musa **al-Khwarizmi**.

He wrote a fundamental treatise on the "Hindu–Arabic numeral system" which was translated into Latin during the 12th century.

Here: al-Khwarizmi was translated into Algorizmi

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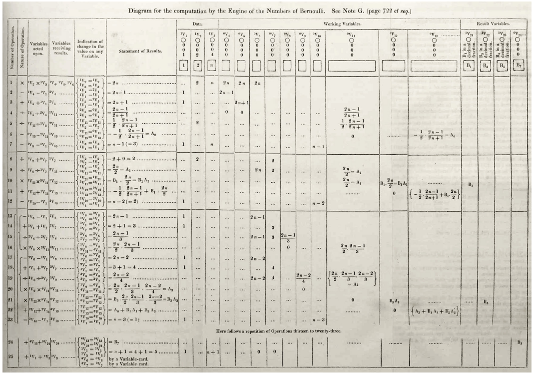

(1815-1852) English mathematician

The first computer program was written by Ada Lovelace for the “Analytical Engine” [design for a simple mechanical computer] by Charles Babbage to compute Bernoulli numbers.

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The formal definition of algorithm goes back to Alan Turing who designed Turing machines as a theoretical concept to simulate the operating principles of a computer (central processing unit = CPU)

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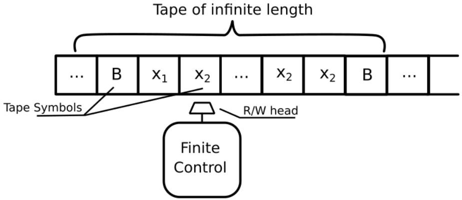

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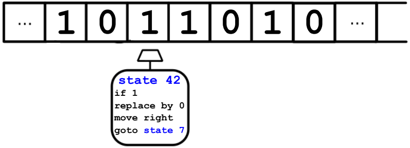

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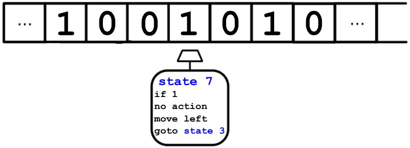

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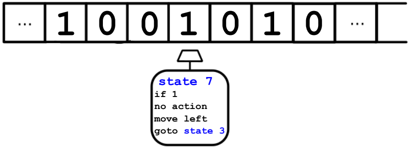

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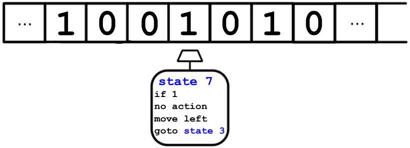

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fixed number of registers $R_1, \ldots, R_k$

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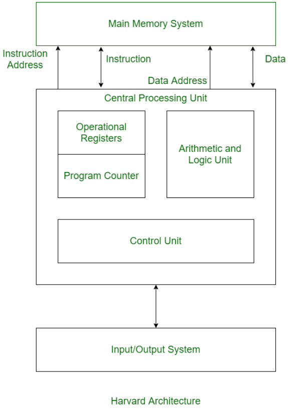
memory and registers store w-bit integers $n \in \{0, \ldots, 2^w - 1\}$



Harvard Architecture

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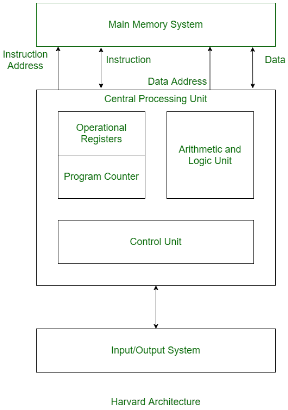

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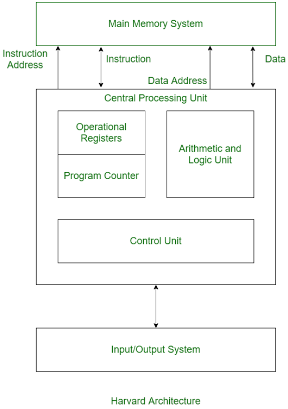

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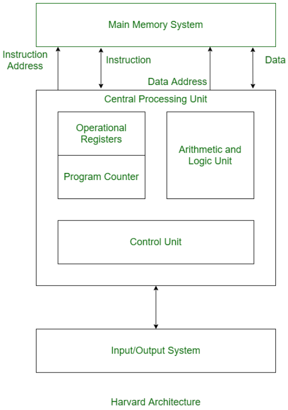

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costs = number of executed step-by-step instructions (i.e., each instruction takes constant time)

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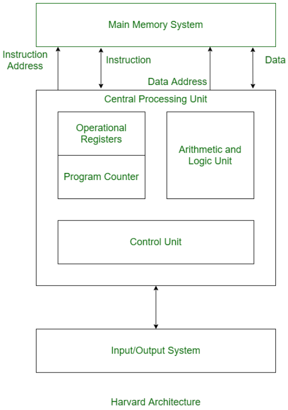

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Already simplified, but typical RAM-code (here for computing $\sum_{i=1}^{n} i$)

```
READ n               # Read the value of n from input
SET R_sum = 0        # Initialize a register R_sum to store the sum
SET R_count = 1      # Initialize a register R_count to store the counter variable

LOOP_START:
    COMPARE R_count > n
    IF_TRUE jump to END_LOOP  # Check if R_count is greater than n

    # Add the current value of R_count to sum
    LOAD R_tmp, R_sum   # Load the current value of sum into a temporary register
    ADD R_tmp, R_count  # Add the value of counter variable to the temporary register "R_tmp += R_count"
    STORE R_sum, R_tmp  # Store the result back in the sum register

    # Increment the counter
    LOAD R_tmp2,  R_count     # Load the current value of R_count into another temporary register
    ADD R_tmp2, 1             # Increment the value in the temporary register "R_tmp2 += 1"
    STORE R_count, R_tmp2     # Store the result back in the R_count register

    jump to LOOP_START  # Go back to the beginning of the loop

END_LOOP:
PRINT R_sum            # Output the final sum
```

Example: Board

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Strictly speaking, we should precisely define the instructions of the RAM model and their costs (which is quite cumbersome!). Nevertheless, the latter should give you a general idea of the term “algorithm” and we provide here a "rough, approximate" definition:

A procedure is an algorithm if . . .

- . . . it is a finite linear sequence of instructions (Instruction 1, followed by Instruction 2 and so on . . .)
- . . . its input is a finite set of values
- . . . in every step only a finite amount of memory is used
- . . . has some finite set of values as output (in case it terminates)

instructions	Example
basic arithmetics	addition, subtraction, multiplication, division, floor, ceiling, modulo,. . .
data movement (save/load/copy)	init array A of size n and save number k at its i-entry ($A[i] := k$)
conditional execution	`IF` (condition) `THEN` Instruction `ELSE` Some-Other-Instruction
jump	`GOTO` instruction x (or line x)
iteration	`WHILE` (condition) `DO` Instruction `REPEAT` Instruction `UNTIL` (condition) `FOR` ($j = 1$ to $n - 1$) `DO` Instruction
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Compute $total\_sum = \sum_{i=1}^{n} i$ for a given integer n.

Verbal "*We define total_sum to be* 0 *and then add to total_sum the integer* 1*, then we add* 2*, . . . , then we add n.*"

for communication of ideas often sufficient, usually indicates only implicitly sequence of instructions
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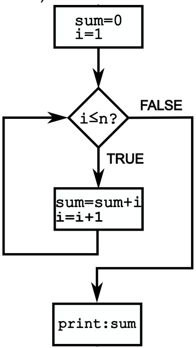

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Some "real" programming language (here python)

```python
def sum_up_to_n(n):

 total_sum = 0
 for i in range(1, n + 1):
     total_sum += i

 print(f"The sum of integers from 1 to {n} is: {total_sum}")
```

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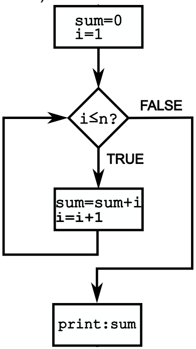

Part 1-2: Correctness of algorithms

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We are, in particular, interested in algorithms that solve problems:

An **instance** of a problem consists of the input (satisfying whatever constraints are imposed in the problem statement) needed to compute a solution to the problem.

Example: Instances of the previous "sum-up" problem are the n for a specific integer (e.g. $n = 3$)

An algorithm is said to be **correct** if, for every input instance, it halts (=terminates) with the correct output w.r.t. the given computational problem. In this case, we also say that the algorithm **solves** the problem.

[$\Longleftrightarrow$ an incorrect algorithm might not halt at all on some input instances, or it might halt with an incorrect answer]

Let's have a look to a specific problem, design an algorithm and show its correctness, i.e., it solves this problem.

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An Example: Sorting Problem

Given: A finite sequence of integers $(a_1, a_2 \ldots, a_n)$
Goal: A re-ordering $(a'_1, a'_2 \ldots, a'_n)$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

$A = (5, 2, 4, 6)$ should become $(2, 4, 5, 6)$

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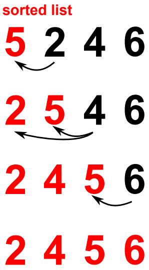

A simple sorting "algorithm" idea:

We assume to have an order list (highlighted in red)

Then, subsequently insert the next element x into this sorted list by comparing x with the elements in sorted list from right to left

We put this into an algorithm, known as `Insertion_Sort`.

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i	×				
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This shows that `Insertion_Sort` correctly works precisely for the input sequence $(5, 2, 4, 6)$.

Part 1-2: Correctness of algorithms

An **instance** of a problem consists of the input (satisfying whatever constraints are imposed in the problem statement) needed to compute a solution to the problem.

Example: Instances of the sorting-problem are all finite sequences of integers, e.g. $(1, 2, 3)$, $(1, 1, 1, \ldots, 1)$, $(5, 4, 5, 3)$, $\ldots$

An algorithm is **correct** or **solves** the problem if, for every input instance, it halts with the correct output w.r.t. the given problem.

So-far, we showed that `Insertion_Sort` correctly works only for the specific instance $(5, 2, 4, 6)$.

Let's proof its correctness, i.e., we show that `Insertion_Sort` terminates and correctly returns a sorted list for any finite input sequence of integers.

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Part 1-2: Correctness of algorithms

Theorem 1

`Insertion_Sort` correctly sorts a given finite sequence A of integers.

Proof.

board via loop-invariants □

Part 1: Summary up to here

We have discussed so-far the topics

1-1 What is an algorithm?

We shortly explained that TMs and equivalent models (RAM) are used to define the term algorithm

Deeper details are beyond the scope of this course. Instead, we provided here a "rough, approximate" definition:

A procedure is an algorithm if ...

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Deeper details are beyond the scope of this course. Instead, we provided here a "rough, approximate" definition:

A procedure is an algorithm if . . .

. . . it is a finite linear sequence of instructions (Instruction 1, followed by Instruction 2 and so on . . .)
. . . its input is a finite set of values
. . . in every step only a finite amount of memory is used
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This finite linear sequence of instructions can be written using e.g. pseudo-code or flowcharts

1-2 Correctness of algorithms

An algorithm is said to be **correct** if, for every input instance, it halts (=terminates) with the correct output w.r.t. the given computational problem. In this case, we also say that the algorithm **solves** the problem.

Parts **1-1** and **1-2** have been examined on a sorting algorithm `Insertion_Sort`.

Let's now continue with Part **1-3** "Runtime"

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We have discussed so-far the topics

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Naive idea: measure the time from start to end in (milli)seconds

say we want to know for some input N how fast the algorithm is:

$N = 4000$ and runtime 6.3 seconds
$N = 8000$ and runtime 51.1 seconds
$N = 16000$ and runtime 410.8 seconds

Hypothesis: For arbitrary N runtime is $\sim 10^{-10}N^3$

not really comparable since this can differ on distinct computers.
we need a notation to classify "runtime" that is independent on the "performance" of computer.

Time complexity

- NOT: measure runtime on a specific computer
- BUT: determine effort for idealized computer model (e.g. RAM-model)
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Add two numbers

	1	1	1	3	7
+		2	8	3	4
=	1	3	9	7	1

Takes 5 single additions.

Hence, addition needs $\max\{m, n\}$ operations (even slightly more if we consider "carryover") for two numbers having m, resp., n digits.

There two main types of cost models:

- the unit-cost model assigns a constant cost to every machine operation, regardless of the size of the numbers involved.
- the logarithmic-cost model, assigns a cost to every machine operation proportional to the number of bits involved [Integer $n \in \{0, \ldots, 2^w - 1\}$ needs w bits to be stored]
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The RAM-model contains instructions commonly found in real computers:

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We denote with $T(|I|)$ the runtime of an algorithm with input I. Here, $|I|$ is the size of the input and $T(|I|)$ is the number of operations/instructions used in this algorithm with input I.

Input $I = A$ with n entries: $|I| = n$

```
Count_Zeros(array A)
    int i, count
    count := 0
    FOR(i := 0 to n − 1)
        IF(A[i] == 0) DO count++
```

variable declaration (e.g. `int` i, $count$):	2
assignment statement (e.g. $i := 0$):	2
increment (i and $count$)	$n + n$
compare "$A[i] == 0$"	n
Σsingle instructions $= T(n) =$	$3n + 4$

Still, this is unsatisfying, e.g. if you have $T(n) = 3n + 4$ vs $T'(n) = 4n$ (which is faster?)

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For $n = 1, 2, 3, 4$ we have $T(n) \geq T'(n)$ and $T(n) < T'(n)$ for $n > 5$

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Σsingle instructions $= T(n) = 4(n-1) + 2\sum_{j=1}^{n-1} t_j + 2\sum_{j=1}^{n-1}(t_j - 1) + 1$

best-case: A already sorted, in which case $A[i] \leq key$ for each $i = j - 1$ and thus, $t_j = 1$ for all j (Line 5,6 not executed).

$T(n) = 4(n-1) + 2\sum_{j=1}^{n-1} 1 + 0 + 1 = 4(n-1) + 2(n-1) + 1 = 6n - 5$ (*linear runtime*)

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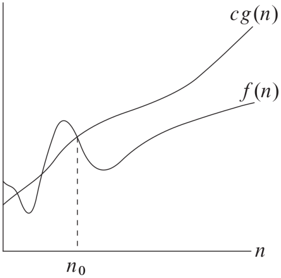

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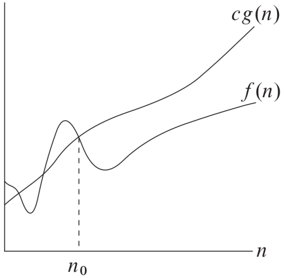

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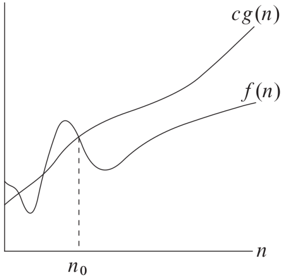

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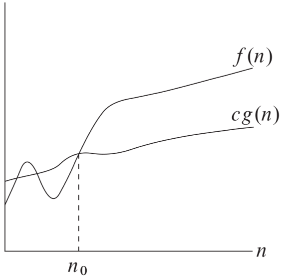

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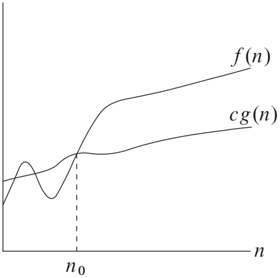

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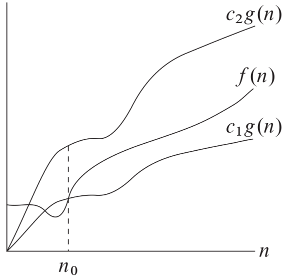

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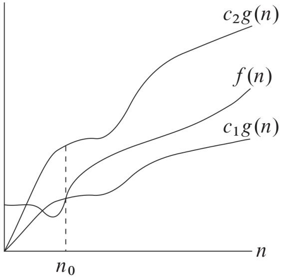

For all $n \geq n_0$, the function $f(n)$ is equal to $g(n)$ to within a constant factor (asymptotically tight bound for $f(n)$)

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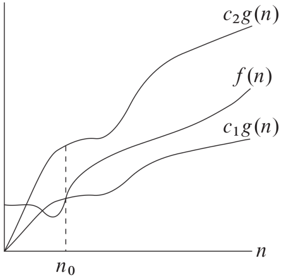

Theorem: $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$ if and only if $f(n) \in \Theta(g(n))$.

[proof exercise]

Part 1-3: Runtime of algorithms (Proofs on board)

For $f(n) = 0.5n^2 + 3n$ show:

$f(n) \in O(n^2)$; $f(n) \in \Omega(n^2)$ (and thus, $f(n) \in \Theta(n^2)$).

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Show $f(n) = 2^{n+1} \in \Theta(2^n)$

For $f(n) = n^2(sin(n))^2 + 50n$ show:

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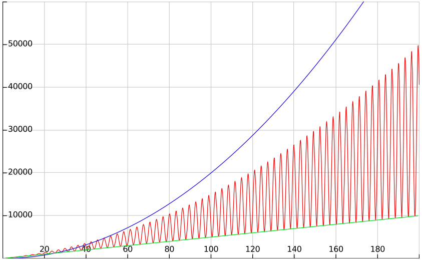

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Insertion-sort revisited:

best-case: $T(n) = 6n - 5$

worst-case: $T(n) = 2n^2 - 1$

Thus, the running time of insertion-sort is in $O(n^2)$, that is, no matter what particular input of size n is chosen, the running time on that input is always bounded from above by some function cn^2 for some constants $c, n_0 > 0$ and all $n \geq n_0$.

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Moreover, these bounds are asymptotically as tight as possible:
The running time of insertion-sort is not $\Omega(n^2)$, since there exists an input for which insertion sort runs in $\Theta(n)$ time (e.g., when the input is already sorted).

It is not contradictory, however, to say that the *worst-case* running time of insertion sort is $\Omega(n^2)$ since there exists an input that causes the algorithm to take $\Omega(n^2)$ time.

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Let $f(n)$ and $g(n)$ be asymptotically positive functions.

Then, for $\Upsilon \in \{O, \Omega, \Theta\}$, it holds that

- $f(n) \in \Upsilon(g(n))$ and $g(n) \in \Upsilon(h(n))$ implies $f(n) \in \Upsilon(h(n))$ [transitivity] — proof board
- $f(n) \in \Upsilon(f(n))$ [reflexivity] — proof exercise

Moreover, it holds that

- $f(n) \in \Theta(g(n))$ if and only if $g(n) \in \Theta(f(n))$ [symmetry] — proof exercise
- $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$ [transpose symmetry] — proof exercise

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Part 1-3: Runtime of algorithms

$O(\ldots)$ (rt=runtime)	typical framework	typical examples
$O(1)$ constant rt	`a=b+c` // `if(a<b)`	assignments, in/output, 32/64bit-arithmetic, cases
$O(\log n)$ logarithmic rt	`while(N>1) N = N/2`	binary search
$O(n)$ linear rt	`for(i=0; i<n; i++){...}`	loop find the maximum
$O(n^2)$ quadratic rt	`for(i=0; i<n; i++)` `for(j=0; j<n; j++) {...}`	double loop, check all pairs

$O(n^3)$ cubic rt	`for(i=0; i<n; i++)`
	`for(j=0; j<n; j++)`
	`for(k=0; k<n; k++) {...}`
		triple loop,
		check all triples

$O(2^n)$ exponential rt	`see combinatorial lecture;)`	exhaustive search check all subsets

Instead of $f(n) \in O(g(n))$ one often writes $f(n) = O(g(n))$ (similar for Ω, Θ)

This is sometimes convenient when establishing certain estimations or calculations.

IMPORTANT NOT !!! $O(g(n)) = f(n)$

Example

$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means that there is some anonymous function $f(n) \in \Theta(n)$ that we do not care to name, such that $2n^2 + 3n + 1 = 2n^2 + f(n)$.

This can help eliminate inessential detail and clutter in an equation and allows us also to write for $\Upsilon \in \{O, \Omega, \Theta\}$:

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Part 1-3: Runtime of algorithms

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Part 1-3: Runtime of algorithms

Exmpl: Applying $O(f(n)) + O(g(n)) = O(\max(f(n), g(n)))$ and $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$

```
Do_Smth(int n)
PRINT "Hello World"
FOR (i = 0 to n − 1) DO
    i := i + 1
    IF (n is even) THEN RETURN 0
    ELSE
        FOR (j = 0 to n − 1) DO
            j := j + 1
```

Exmpl: Applying $O(f(n)) + O(g(n)) = O(\max(f(n), g(n)))$ and $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$

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All basic-instructions (eg. PRINT, $i = 0$, $j := j + 1$, RETURN 0, ...) in $O(1)$ time

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1 PRINT "Hello World"
2 FOR (i = 0 to n − 1) DO
3     i := i + 1
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6        FOR (j = 0 to n − 1) DO
7           j := j + 1
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All basic-instructions (eg. PRINT, $i = 0$, $j := j + 1$, RETURN 0, $\ldots$) in $O(1)$ time

Do_Smth consists of two main-parts:

$$A_1 = \texttt{PRINT "Hello World"} \text{ and } A_2 = \text{Line 2-7}$$

Hence, runtime of DO_SMTH is in $O(1)+$ runtime $A_2 \implies$ examine A_2 !

Part 1-3: Runtime of algorithms

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runtime A_2 = Line 2-7

The most expensive task within the loop in Line 2 is in $O(n)$:

Part 1-3: Runtime of algorithms

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3     i := i + 1
4     IF (n is even) THEN RETURN 0
5     ELSE
6        FOR (j = 0 to n − 1) DO
7           j := j + 1
```

runtime A_2 = Line 2-7

The most expensive task within the loop in Line 2 is in $O(n)$:

Line 3: $O(1)$
Line 4: $O(1) + O(1) = O(\max(1, 1)) = O(1)$
Line 5: $O(1)$
Line 6-7: $O(n) \cdot O(1) = O(n \cdot 1) = O(n)$

Part 1-3: Runtime of algorithms

Exmpl: Applying $O(f(n)) + O(g(n)) = O(\max(f(n), g(n)))$ and $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$

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Do_Smth(int n)
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FOR-loop in Line 2 runs n times. runtime $A_2 = O(n) \cdot O(n) = O(n \cdot n) = O(n^2)$

Part 1-3: Runtime of algorithms

Exmpl: Applying $O(f(n)) + O(g(n)) = O(\max(f(n), g(n)))$ and $O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n))$

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```

Runtime DO_SMTH is in $O(1)+$ runtime A_2 = $O(1) + O(n^2) = O(\max(1, n^2)) = O(n^2)$

Summary up to here

$O(g(n)) := \{f(n)\colon \exists$ constants $c, n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$

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Theorem: $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$ if and only if $f(n) \in \Theta(g(n))$.

Let $f(n)$ and $g(n)$ be asymptotically positive functions and $\Upsilon \in \{O, \Omega, \Theta\}$. Then,

$f(n) \in \Upsilon(g(n))$ and $g(n) \in \Upsilon(h(n))$ implies $f(n) \in \Upsilon(h(n))$ [transitivity]

$f(n) \in \Upsilon(f(n))$ [reflexivity]

Moreover, it holds that

$f(n) \in \Theta(g(n))$ if and only if $g(n) \in \Theta(f(n))$ [symmetry]

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It could be that some of these equations must be proven in exercises or the exam!

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Further example

```
Halve(number n)
    WHILE (n > 1) DO
        n := n/2
```

$T(n) =$

Part 1-3: Runtime of algorithms

Further example

```
Halve(number n)
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$$T(n) = \Theta(1) + T(\frac{n}{2})$$

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$$T(n) = \Theta(1) + T(\frac{n}{2})$$

$$= \Theta(1) + (\Theta(1) + T(\frac{n}{4})) = 2 \cdot \Theta(1) + T(\frac{n}{2^2})$$

Part 1-3: Runtime of algorithms

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How often can one repeat this, that is, what is N?

Part 1-3: Runtime of algorithms

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How often can one repeat this, that is, what is N?

In other words: For which $k = \frac{n}{2^N}$ does `Halve`(k) terminate?

Answer: For any $k \leq 1$

Further example

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Answer: For any $k \leq 1 \iff \frac{n}{2^N} \leq 1 \iff n \leq 2^N \iff \log_2(n) \leq N$

Further example

`Halve`(number n)

    WHILE ($n > 1$) DO
        $n := \frac{n}{2}$

$$
\begin{aligned}
T(n) &= \Theta(1) + T(\frac{n}{2}) \\
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Part 1-3: Runtime of algorithms

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$$= \Theta(\max\{\log_2(n), 1\}) = \Theta(\log_2(n))$$

Iterative vs. recursive algorithms

iterative	recursive
Sum(n) $total\_sum := 0$ FOR ($i = 1$ to n) DO $total\_sum := total\_sum + i$ PRINT $total\_sum$	Sum(int n) IF($n = 1$) THEN RETURN 1 RETURN $n +$ SUM$(n - 1)$

Iterative vs. recursive algorithms

iterative	recursive

Sum(n)
  $total\_sum := 0$
  FOR ($i = 1$ to n) DO
    $total\_sum := total\_sum + i$
  PRINT $total\_sum$
	Sum(int n)
	  IF($n = 1$) THEN RETURN 1
	  RETURN $n + \text{SUM}(n-1)$

What are these algorithms doing?

Iterative vs. recursive algorithms

iterative	recursive

`Sum`(n)
  $total\_sum := 0$
  `FOR` ($i = 1$ to n) `DO`
    $total\_sum := total\_sum + i$
  `PRINT` $total\_sum$
	`Sum`(int n)
	  `IF`($n = 1$) `THEN RETURN` 1
	  `RETURN` $n + \text{SUM}(n-1)$

What are these algorithms doing? **Answer:** `Sum` computes the sum $\sum_{i=1}^{n} i$, where $n \geq 1$.

Iterative vs. recursive algorithms

iterative	recursive
Sum(n) $total\_sum := 0$ FOR ($i = 1$ to n) DO $total\_sum := total\_sum + i$ PRINT $total\_sum$	Sum(int n) IF($n = 1$) THEN RETURN 1 RETURN $n +$ SUM$(n - 1)$

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Iterative vs. recursive algorithms

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Iterative vs. recursive algorithms

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Iterative vs. recursive algorithms

iterative	recursive

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$total\_sum := 1 + 2 = 3$	RETURN $2 + \text{SUM}(1)$ (the return value of SUM(2))
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Iterative vs. recursive algorithms

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$total\_sum := 6 + 4 = 10$	$\Longrightarrow$ $2 +$ SUM$(1) = 2 + 1 = 3$ [SUM(2) = 3]

Iterative vs. recursive algorithms

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`Sum`(n) $total\_sum := 0$ `FOR` ($i = 1$ to n) `DO` $total\_sum := total\_sum + i$ `PRINT` $total\_sum$	`Sum`(int n) `IF`($n = 1$) `THEN RETURN` 1 `RETURN` $n + \text{SUM}(n-1)$

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Runtime iterative SUM: $\Theta(n)$ [Exercise]

Part 1-3: Runtime of algorithms

Iterative vs. recursive algorithms

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`Sum(n)`
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$$T(n) = \Theta(1) + T(n-1)$$

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\begin{aligned}
T(n) &= \Theta(1) + T(n-1) \\
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\end{aligned}
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Iterative vs. recursive algorithms

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T(n) &= \Theta(1) + T(n-1) \\
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&= \ldots \\
&= (n-1)\Theta(1) + T(1) \in \Theta(n) \text{ since } T(1) \in \Theta(1)
\end{aligned}
$$

Part 1-3: Runtime of algorithms

Often recurrences come in the form

$$T(n) = aT(n/b) + f(n)$$

with constants $a \geq 1$ and $b > 1$.

$n \in \mathbb{N}_{\geq 1}$ is the input size

a is the number of subproblems in the recursion

n/b is the size of a single subproblem and means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$

$f(n)$ denotes the cost incurred by dividing the problem and combining the partial solutions

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Example:

```
Some_Rec(n)
  IF (n > 1) THEN
     someTask
     Some_Rec(n/3) + Some_Rec(n/3)
```

Suppose someTask has runtime in $\Theta(n^5)$

$\implies T(n) = 2T(n/3) + \Theta(n^5)$, $a = 2, b = 3, d = 5$

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Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_2 n) & \text{if } a = b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

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$\Longrightarrow \; a < b^d \; \Longrightarrow \; T(n) = \Theta(n^5)$

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Example:

```
Halve(n)
    IF (n > 1) THEN Halve(n/2)
```

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Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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Runtime without Master Theorem: $O(\log_2(n))$ (similar arguments as for `Halve` above with `WHILE`-loop)

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Runtime without Master Theorem: $O(\log_2(n))$ (similar arguments as for `Halve` above with `WHILE`-loop)

With Master Theorem: $T(n) = T(n/2) + \Theta(1)$

$\implies a = 1, b = 2, d = 0$

In formula above: $1 = a = b^d = 2^0$ and thus, runtime is in $\Theta(n^d \log_2 n) = \Theta(n^0 \log_2 n) = \Theta(\log_2 n)$

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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Further examples: Assume that $d = 2$ and $b = 3$:

$a = 8$: $T(n) = 8T(\frac{n}{3}) + \Theta(n^2) \xrightarrow{8<3^2} T(n) = \Theta(n^2)$

$a = 9$: $T(n) = 9T(\frac{n}{3}) + \Theta(n^2) \xrightarrow{9=3^2} T(n) = \Theta(n^2 \log_2 n)$

$a = 10$: $T(n) = 10T(\frac{n}{3}) + \Theta(n^2) \xrightarrow{10>3^2} T(n) = \Theta(n^{log_3(10)})$

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_2 n) & \text{if } a = b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

proof: For simplicity write $n^d = \Theta(n^d)$

$$\begin{aligned} T(n) &= aT(\tfrac{n}{b}) + n^d \quad \text{//use formular for input of size } n/b \\ &= a[aT(\tfrac{n}{b^2}) + (\tfrac{n}{b})^d] + n^d = a^2 T(\tfrac{n}{b^2}) + a(\tfrac{n}{b})^d + n^d \quad \text{//use formular for input of size } n/b^2 \\ &= a^2[aT(\tfrac{n}{b^3}) + (\tfrac{n}{b^2})^d] + a(\tfrac{n}{b})^d + n^d = a^3 T(\tfrac{n}{b^3}) + a^2(\tfrac{n}{b^2})^d + a(\tfrac{n}{b})^d + n^d \\ &= \ldots \\ &= a^k T(\tfrac{n}{b^k}) + \textstyle\sum_{j=0}^{k-1} a^j \left(\tfrac{n}{b^j}\right)^d \quad \text{//by induction (exercise)} \\ &= a^k T(\tfrac{n}{b^k}) + n^d \textstyle\sum_{j=0}^{k-1} \left(\tfrac{a}{b^d}\right)^j \end{aligned}$$

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$$T(n) = aT(\tfrac{n}{b}) + n^d \quad \text{//use formular for input of size } n/b$$

$$= a[aT(\tfrac{n}{b^2}) + (\tfrac{n}{b})^d] + n^d = a^2 T(\tfrac{n}{b^2}) + a(\tfrac{n}{b})^d + n^d \quad \text{//use formular for input of size } n/b^2$$

$$= a^2[aT(\tfrac{n}{b^3}) + (\tfrac{n}{b^2})^d] + a(\tfrac{n}{b})^d + n^d = a^3 T(\tfrac{n}{b^3}) + a^2(\tfrac{n}{b^2})^d + a(\tfrac{n}{b})^d + n^d$$

$$= \ldots$$

$$= a^k T(\tfrac{n}{b^k}) + \sum_{j=0}^{k-1} a^j \left(\tfrac{n}{b^j}\right)^d \quad \text{//by induction (exercise)}$$

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$$= a^2[aT(\tfrac{n}{b^3}) + (\tfrac{n}{b^2})^d] + a(\tfrac{n}{b})^d + n^d = a^3T(\tfrac{n}{b^3}) + a^2(\tfrac{n}{b^2})^d + a(\tfrac{n}{b})^d + n^d$$

$$= \dots$$

$$= a^kT(\tfrac{n}{b^k}) + \sum_{j=0}^{k-1} a^j \left(\tfrac{n}{b^j}\right)^d \quad \text{//by induction (exercise)}$$

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Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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Part 1-3: Runtime of algorithms

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$T(1)$ terminates.

Hence, it terminates for k for which $T(1) = T(\frac{n}{b^k})$ holds

$\iff 1 = \frac{n}{b^k} \iff b^k = n \iff k = \log_b(n)$

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Since $T(\frac{n}{b^{\log_b(n)}}) = T(1)$ we have: $T(n) = \Theta(a^{\log_b(n)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Since $a^{\log_b(n)} = n^{\log_b(a)}$ (exercise!), we have: $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

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We consider now the three cases: $a < b^d$, $a = b^d$ and $a > b^d$

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$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \geq \left(\frac{a}{b^d}\right)^0 = 1 \implies \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Omega(1)$

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Case $a < b^d$:

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \geq \left(\frac{a}{b^d}\right)^0 = 1 \implies \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Omega(1)$$

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{a}{b^d}\right)^j \overset{\text{geom. series}}{=} \frac{1}{1-\frac{a}{b^d}} = O(1) \text{ //since } \frac{a}{b^d} \in (0,1) \text{ and constant}$$

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$$

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_2 n) & \text{if } a = b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

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Case $a < b^d$:

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \geq \left(\frac{a}{b^d}\right)^0 = 1 \implies \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Omega(1)$$

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{a}{b^d}\right)^j \overset{\text{geom. series}}{=} \frac{1}{1-\frac{a}{b^d}} = O(1) \text{ //since } \frac{a}{b^d} \in (0,1) \text{ and constant}$$

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Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$$

Master Theorem [simplified version]

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Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \geq \left(\frac{a}{b^d}\right)^0 = 1 \implies \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Omega(1)$$

$$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{a}{b^d}\right)^j \overset{\text{geom. series}}{=} \frac{1}{1-\frac{a}{b^d}} = O(1)$$ //since $\frac{a}{b^d} \in (0,1)$ and constant

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Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$$

$$= \Theta(n^{\log_b(a)}) + n^d \Theta(1)$$

$$= \Theta(n^{\log_b(a)} + n^d)$$

Since $a < b^d$ we have $\log_b(a) \leq \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d + n^d) = \Theta(n^d)$ as desired

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$$

$$= \Theta(n^{\log_b(a)}) + n^d \Theta(1)$$

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Since $a < b^d$ we have $\log_b(a) \leq \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d + n^d) = \Theta(n^d)$ as desired

Part 1-3: Runtime of algorithms

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_2 n) & \text{if } a = b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$$

$$= \Theta(n^{\log_b(a)}) + n^d \Theta(1)$$

$$= \Theta(n^{\log_b(a)} + n^d)$$

Since $a < b^d$ we have $\log_b(a) \leq \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d + n^d) = \Theta(n^d)$ as desired

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log_2 n) & \text{if } a = b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$$

$$= \Theta(n^{\log_b(a)}) + n^d \Theta(1)$$

$$= \Theta(n^{\log_b(a)} + n^d)$$

Since $a < b^d$ we have $\log_b(a) \leq \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d + n^d) = \Theta(n^d)$ as desired

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$$

$$= \Theta(n^{\log_b(a)}) + n^d \Theta(1)$$

$$= \Theta(n^{\log_b(a)} + n^d)$$

Since $a < b^d$ we have $\log_b(a) \leq \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d + n^d) = \Theta(n^d)$ as desired

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a < b^d$: $\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \in \Theta(1)$

$$\begin{aligned} T(n) &= \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j \\ &= \Theta(n^{\log_b(a)}) + n^d \Theta(1) \\ &= \Theta(n^{\log_b(a)} + n^d) \end{aligned}$$

Since $a < b^d$ we have $\log_b(a) \leq \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d + n^d) = \Theta(n^d)$ as desired

Master Theorem [simplified version]

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a = b^d$: $\iff \frac{a}{b^d} = 1$

$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j = \sum_{j=0}^{\log_b(n)-1} 1 = \log_b(n)$

$T(n) = \Theta(n^{\log_b(a)}) + n^d \log_b(n)$

Since $a = b^d$ we have $\log_b(a) = \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d) + n^d \log_b(n) = \Theta(n^d \log_b(n))$

Since $\log_2(n) = \frac{\log_b(n)}{\log_b(2)}$ and $\log_b(2)$ is constant, we have $\Theta(\log_b(n)) = \Theta(\log_2(n))$ and thus,

$T(n) = \Theta(n^d \log_2(n))$ as desired

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Case $a = b^d$: $\iff \frac{a}{b^d} = 1$

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Since $a = b^d$ we have $\log_b(a) = \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d) + n^d \log_b(n) = \Theta(n^d \log_b(n))$

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a = b^d$: $\iff \frac{a}{b^d} = 1$

$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j = \sum_{j=0}^{\log_b(n)-1} 1 = \log_b(n)$

$T(n) = \Theta(n^{\log_b(a)}) + n^d \log_b(n)$

Since $a = b^d$ we have $\log_b(a) = \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d) + n^d \log_b(n) = \Theta(n^d \log_b(n))$

Since $\log_2(n) = \frac{\log_b(n)}{\log_b(2)}$ and $\log_b(2)$ is constant, we have $\Theta(\log_b(n)) = \Theta(\log_2(n))$ and thus,

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Case $a = b^d$: $\iff \frac{a}{b^d} = 1$

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Since $a = b^d$ we have $\log_b(a) = \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d) + n^d \log_b(n) = \Theta(n^d \log_b(n))$

Since $\log_2(n) = \frac{\log_b(n)}{\log_b(2)}$ and $\log_b(2)$ is constant, we have $\Theta(\log_b(n)) = \Theta(\log_2(n))$ and thus,

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Case $a = b^d$: $\iff \frac{a}{b^d} = 1$

$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j = \sum_{j=0}^{\log_b(n)-1} 1 = \log_b(n)$

$T(n) = \Theta(n^{\log_b(a)}) + n^d \log_b(n)$

Since $a = b^d$ we have $\log_b(a) = \log_b(b^d) = d$ and therefore, $T(n) = \Theta(n^d) + n^d \log_b(n) = \Theta(n^d \log_b(n))$

Since $\log_2(n) = \frac{\log_b(n)}{\log_b(2)}$ and $\log_b(2)$ is constant, we have $\Theta(\log_b(n)) = \Theta(\log_2(n))$ and thus,

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Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

Case $a = b^d$: $\iff \frac{a}{b^d} = 1$

$\sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j = \sum_{j=0}^{\log_b(n)-1} 1 = \log_b(n)$

$T(n) = \Theta(n^{\log_b(a)}) + n^d \log_b(n)$

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Since $\log_2(n) = \frac{\log_b(n)}{\log_b(2)}$ and $\log_b(2)$ is constant, we have $\Theta(\log_b(n)) = \Theta(\log_2(n))$ and thus,

$T(n) = \Theta(n^d \log_2(n))$ as desired

Part 1-3: Runtime of algorithms

Master Theorem [simplified version]

Let $a \geq 1$, $b > 1$ and $d \geq 0$ be constants and $n \in \mathbb{N}_{\geq 1}$. If $T(n) = aT(n/b) + \Theta(n^d)$, then

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proof: For simplicity write $n^d = \Theta(n^d)$ and we have $T(n) = \Theta(n^{\log_b(a)}) + n^d \sum_{j=0}^{\log_b(n)-1} \left(\frac{a}{b^d}\right)^j$

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Part 1-3: Time Complexity & Space Complexity

Space complexity is a measure of the amount of working storage an algorithm needs and is also often expressed asymptotically in big-O, Big-Ω, Big-Θ notation.

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Requires 3 units of space for the parameters x, y, z and 1 for the local variable r.

Space complexity is in $O(1)$

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Requires n units of space for array a and 2 for the local variables r and i.

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Fact_iter(int n)
  int fac = 1
  FOR (i = 1 to n) DO
      fac := fac · i
  RETURN fac
```

```
Fact_rec(int n)
  IF(n == 0 or n == 1) THEN
      RETURN 1
  ELSE
      RETURN n · Fact_rec(n − 1)
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⋮

for 2 the return-value Fact_rec(1) must temporarily be stored
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At this point the values can be used to compute Fact_rec(n) and we temporarily stored $n-1 = O(n)$ variables.

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Side Note:
During each time step, you can only access one memory location. Therefore you can never access more memory locations than you have time
$\implies$ space complexity is bounded by time complexity

Does runtime matter?

	insertion-sort	merge-sort [later]
runtime	$O(n^2)$	$O(n\log_2(n))$

Should we care about factor n vs $\log_2(n)$?

For large enough n and constant c, we have

	insertion-sort	merge-sort
runtime	$c \cdot n^2$	$c \cdot n\log_2(n)$

Say $c = 100$ and $n = 10^7$ (e.g. list of population in Sweden). Suppose we have a computer that can perform $10^9 op/s$ where op/s = operations per seconds.

insertion-sort: $\frac{100\cdot(10^7)^2 op}{10^9 op/s} = \frac{10^{16} op}{10^9 op/s} = 10^7 s$

merge-sort: $\frac{100\cdot 10^7 \cdot \log_2(10^7) op}{10^9 op/s} = \frac{10^9 \cdot \log_2(10^7) op}{10^9 op/s} = \log_2(10^7) s$

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For large enough n and constant c, we have

	insertion-sort	merge-sort
runtime	$c \cdot n^2$	$c \cdot n\log_2(n)$

Say $c = 100$ and $n = 10^7$ (e.g. list of population in Sweden). Suppose we have a computer that can perform $10^9 op/s$ where op/s = operations per seconds.

insertion-sort: $\frac{100\cdot(10^7)^2 op}{10^9 op/s} = \frac{10^{16} op}{10^9 op/s} = 10^7 s$

merge-sort: $\frac{100\cdot 10^7\cdot\log_2(10^7) op}{10^9 op/s} = \frac{10^9\cdot\log_2(10^7) op}{10^9 op/s} = \log_2(10^7)s$

Runtime matters !

Part 1-3: Runtime

Does runtime matter?

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Runtime matters !

Part 1-4: Elementary Data Structures

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The right organizational form and choice of data structure significantly impact the efficiency of data operations.

Example:
Consider a phone book. There it is easy to find a phone number for a given name based on the alphabetical order.
What if we are interested in the reverse task (finding for a given number the name)?
Ideas?

The optimal choice of a data structure is not always obvious and one data structure might be very suitable for one task but not for some other

Determining an efficient data structure is usually influenced by the operations needed later on the data (searching, replacing, re-sorting, . . .)

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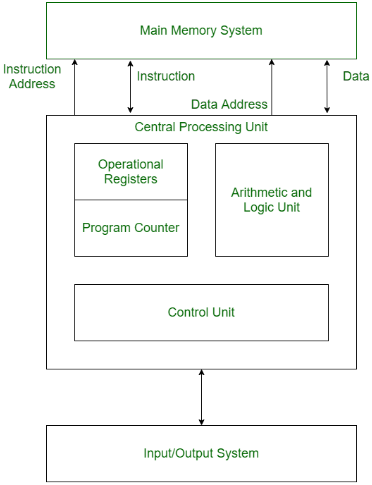
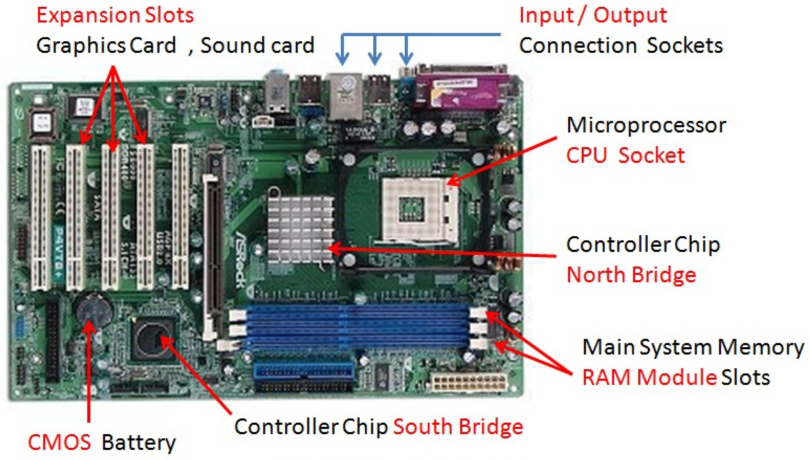
Part 1-4: Elementary Data Structures

How does memory work and how is this related to Data Structures?



Harvard Architecture

How does memory work and how is this related to Data Structures?



mainboard of a computer

Part 1-4: Elementary Data Structures

How does memory work and how is this related to Data Structures?

main memory or also RAM = Random Access Memory

How does memory work and how is this related to Data Structures?

Main memory consists of a number of regularly arranged memory cells, comparable to the compartments of a cabinet.

How does memory work and how is this related to Data Structures?

Since memory cells are regularly arranged, they can be numbered consecutively.
Each cell therefore has a unique number (=address).

How does memory work and how is this related to Data Structures?

All memory cells are the same size and can store a value (number, character, . . .).
This value is a fixed-length sequence of 0s and 1s (e.g. 1byte = 8 bits)

[8 bit per cell is pure convention (a few exceptions exist)].

How does memory work and how is this related to Data Structures?

The value stored in a cell represents some information (e.g. a number or a character)

How does memory work and how is this related to Data Structures?

But also "longer" information can be stored using "chunks of cells"

(e.g., the first 8bits of a 32bit integer *n* [to store *n* we need then 4 cells each of size 1byte]
or the first 3 characters of the alphabet)

How does memory work and how is this related to Data Structures?

But also "longer" information can be stored using "chunks of cells"

Difference between a 32-bit and a 64-bit architecture? *n*-bit architecure means that CPU can handle data in chunks of *n*-bit at a time. Thus, *n*-bit computer can can process data and perform calculations on numbers that are *n*-bits long.

32-bit system that can access 2^{32} (or 4,294,967,296) bytes of RAM. Meanwhile, a 64-bit processor can handle 2^{64} (or 18,446,744,073,709,551,616) bytes of RAM. In other words, a 64-bit processor can process more data than 4 billion 32-bit processors combined.

Part 1-4: Elementary Data Structures

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A variable in (compiled) source-code refers to one or more consecutive cells in memory that store the "value/information" we assigned to this variable.

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A variable in (compiled) source-code refers to one or more consecutive cells in memory that store the "value/information" we assigned to this variable.

Variables can thus contain values or be pointers to another variable.

Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

Adr.	.
	.
	.
6	–
7	–
8	–
	.
	.
80	–
	.

```
int x // init integer variable x
int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
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```

Memory "somewhat similar to what `Python` does"

```
x = 5 // init cell for x and 5 and x contains address of 5
y = 10 // init cell for y and 10 and y contains address of 10
print(x) // prints "5"
print(y) // prints "10"
x=y // let x "point to" address that y "points to"
```

As "5" is no longer used, memory cell 21 is freed up [garbage collector].

```
print(x) // prints "10"
y=42 // y contains address of 42
print(x) // prints "10"
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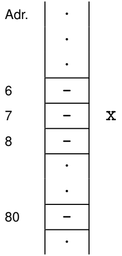
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`	Memory "somewhat similar to what `Python` does"

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Storage of information in different languages (here as example `C` / `Python`)

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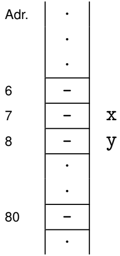


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Memory			C
Adr.	.		
	.		
	.		
6	–		
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8	–	y	
	.		
	.		
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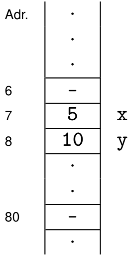
Memory `C`

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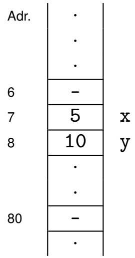


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Memory `C`

Adr.	Memory	
	.	
	.	
	.	
6	–	
7	5	x
8	10	y
	.	
	.	
80	–	
	.	

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Part 1-4: Elementary Data Structures

Storage of information in different languages (here as example `C` / `Python`)

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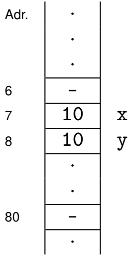


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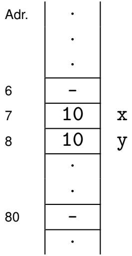
y=42 // y contains address of 42
print(x) // prints "10"
print(y) // prints "42"
```

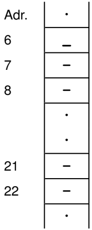
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`




```
int x // init integer variable x
int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
printf("%i", x) // prints "5"
printf("%i", y) // prints "10"
printf("%p", &x) // prints "7" (address of x)
x = y
printf("%i", x) // prints "10"
printf("%p", &x) // prints "7" (address of x)
int *px; // init pointer px that point to some integer variable
px = &x;
printf("%p", px); // prints "7" (content of px)
printf("%i", *px); // prints "10" (content of content of px [Dereference])
```

Memory "somewhat similar to what `Python` does"



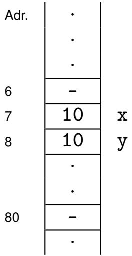

```
x = 5 // init cell for x and 5 and x contains address of 5
y = 10 // init cell for y and 10 and y contains address of 10
print(x) // prints "5"
print(y) // prints "10"
x=y // let x "point to" address that y "points to"
```

As "5" is no longer used, memory cell 21 is freed up [garbage collector].

```
print(x) // prints "10"
y=42 // y contains address of 42
print(x) // prints "10"
print(y) // prints "42"
```

Storage of information in different languages (here as example `C` / `Python`)

Memory `C`




```
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int y // init integer variable y
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y = 10 // assign 5 to y
printf("%i", x) // prints "5"
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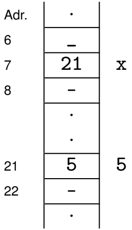


```
x = 5 // init cell for x and 5 and x contains address of 5
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print(y) // prints "10"
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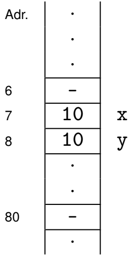
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

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printf("%i", y) // prints "10"
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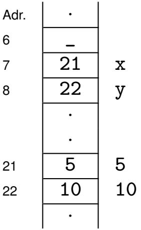
Memory "somewhat similar to what `Python` does"




```
x = 5 // init cell for x and 5 and x contains address of 5
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print(y) // prints "10"
x=y // let x "point to" address that y "points to"
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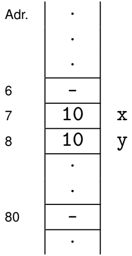
Memory `C`

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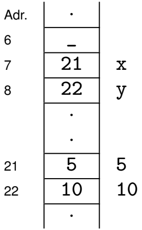


```
x = 5 // init cell for x and 5 and x contains address of 5
y = 10 // init cell for y and 10 and y contains address of 10
print(x) // prints "5"
print(y) // prints "10"
x=y // let x "point to" address that y "points to"
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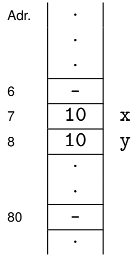
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`




```
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int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
printf("%i", x) // prints "5"
printf("%i", y) // prints "10"
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x = y
printf("%i", x) // prints "10"
printf("%p", &x) // prints "7" (address of x)
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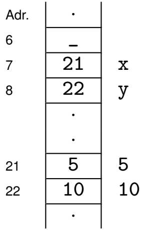


```
x = 5 // init cell for x and 5 and x contains address of 5
y = 10 // init cell for y and 10 and y contains address of 10
print(x) // prints "5"
print(y) // prints "10"
x=y // let x "point to" address that y "points to"
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As "5" is no longer used, memory cell 21 is freed up [garbage collector].

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print(x) // prints "10"
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print(y) // prints "42"
```

Storage of information in different languages (here as example C / Python)

Memory C

Adr.		
	.	
	.	
	.	
6	–	
7	10	x
8	10	y
	.	
	.	
80	–	
	.	

```
int x // init integer variable x
int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
printf("%i", x) // prints "5"
printf("%i", y) // prints "10"
printf("%p", &x) // prints "7" (address of x)
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printf("%i", *px); // prints "10" (content of content of px [Dereference])
```

Memory "somewhat similar to what Python does"

Adr.	.	
6	–	
7	22	x
8	22	y
	.	
	.	
21	5	
22	10	10
	.	

```
x = 5 // init cell for x and 5 and x contains address of 5
y = 10 // init cell for y and 10 and y contains address of 10
print(x) // prints "5"
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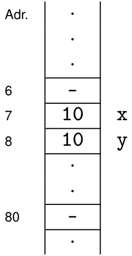
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

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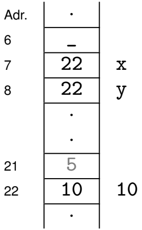
Memory "somewhat similar to what `Python` does"




```
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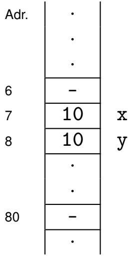
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

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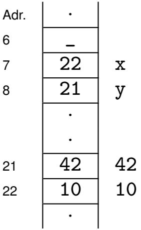
Memory "somewhat similar to what `Python` does"




```
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Storage of information in different languages (here as example `C` / `Python`)

Memory — `C`

Adr.		
	.	
	.	
	.	
6	–	
7	10	x
8	10	y
	.	
	.	
80	–	
	.	

```
int x // init integer variable x
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```

Memory — "somewhat similar to what `Python` does"

Adr.		
	.	
6	–	
7	22	x
8	21	y
	.	
	.	
21	42	42
22	10	10
	.	

```
x = 5 // init cell for x and 5 and x contains address of 5
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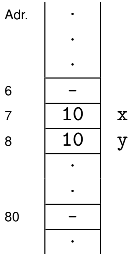
Part 1-4: Elementary Data Structures

Storage of information in different languages (here as example C / Python)

Memory C	Memory "somewhat similar to what Python does"

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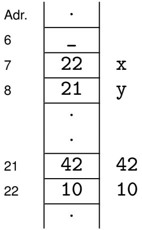


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print(x) // prints "10"
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Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

Adr.	.	
	.	
	.	
6	–	
7	10	x
8	10	y
	.	
	.	
80		
	.	

```
int x // init integer variable x
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x = 5 // assign 5 to x
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Memory "somewhat similar to what `Python` does"

Adr.	.	
6	–	
7	22	x
8	21	y
	.	
	.	
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	.	

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In `python` there are lot of secrets in the memory allocation that cannot directly be handled by user and a lot of vodoo (incl. garbage collection) takes control about the latter

Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

Adr.		
	.	
	.	
	.	
6	–	
7	10	x
8	10	y
	.	
	.	
80	7	px
	.	

```
int x // init integer variable x
int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
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Memory "somewhat similar to what `Python` does"

Adr.		
	.	
6	–	
7	22	x
8	21	y
	.	
	.	
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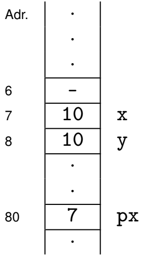
Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

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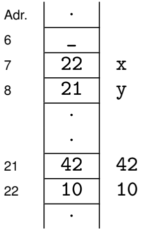
Memory "somewhat similar to what `Python` does"




```
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print(y) // prints "10"
x=y // let x "point to" address that y "points to"
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Storage of information in different languages (here as example `C` / `Python`)

Memory `C`

Adr.	Memory	Variable
	.	
	.	
	.	
6	–	
7	10	x
8	10	y
	.	
	.	
80	7	px
	.	

```
int x // init integer variable x
int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
printf("%i", x) // prints "5"
printf("%i", y) // prints "10"
printf("%p", &x) // prints "7" (address of x)
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```

Memory "somewhat similar to what `Python` does"

Adr.	Memory	Variable
	.	
6	–	
7	22	x
8	21	y
	.	
	.	
21	42	42
22	10	10
	.	

```
x = 5 // init cell for x and 5 and x contains address of 5
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Part 1-4: Elementary Data Structures

Storage of information in different languages (here as example `C` / `Python`)

Memory — C

Adr.		
	.	
	.	
	.	
6	–	
7	10	x
8	10	y
	.	
	.	
80	7	px
	.	

```
int x // init integer variable x
int y // init integer variable y
x = 5 // assign 5 to x
y = 10 // assign 5 to y
printf("%i", x) // prints "5"
printf("%i", y) // prints "10"
printf("%p", &x) // prints "7" (address of x)
x = y
printf("%i", x) // prints "10"
printf("%p", &x) // prints "7" (address of x)
int *px; // init pointer px that point to some integer variable
px = &x;
printf("%p", px); // prints "7" (content of px)
printf("%i", *px); // prints "10" (content of content of px [Dereference])
```

many famous games are based on game engines written in C/C++ (Fortnite, GTA, DOOM, Civilization,. . .)

Memory — "somewhat similar to what `Python` does"

Adr.		
	.	
6	–	
7	22	x
8	21	y
	.	
	.	
21	42	42
22	10	10
	.	

```
x = 5 // init cell for x and 5 and x contains address of 5
y = 10 // init cell for y and 10 and y contains address of 10
print(x) // prints "5"
print(y) // prints "10"
x=y // let x "point to" address that y "points to"
As "5" is no longer used, memory cell 21 is freed up [garbage collector].
print(x) // prints "10"
y=42 // y contains address of 42
print(x) // prints "10"
print(y) // prints "42"
```

In `python` there are lot of secrets in the memory allocation that cannot directly be handled by user and a lot of vodoo (incl. garbage collection) takes control about the latter

Part 1-4: Elementary Data Structures

Pointer = variable `p` that stores address of another memory cell containing information about "some object *x*".

in symbols "$\mathtt{p} \rightarrow x$"

Data structures can be classified as either contiguous or linked, depending upon whether they are based on arrays or pointers:

- **Contiguously-allocated structures** are composed of single slabs of memory, and include arrays, matrices, heaps, and hash tables.
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Analogy/Example:
Init new array L of length 3 [=allocate 3 consecutive cells (here the ones with address 13,14,15)] and put $L[1] = a$, $L[2] = b$, $L[3] = c$

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Advantages:

Constant-time access given the index

Because the index of each element maps directly to a particular memory address, we can access arbitrary data items instantly provided we know the index.

Space efficiency

Arrays consist purely of data, so no space is wasted with links or other formatting information.

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Disadvantages:

Fixed size and content

An array can only save one type of data (e.g. only integer, or only bool, ...)

One cannot adjust the size of an array in the middle of a program's execution

Our program will fail soon as we try to add an $(n+1)$-entry if only space for n records was allocated (= overflow). This can be compensated by allocating extremely large arrays, but this can waste space.

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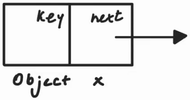
A **(single) linked list** is a data structure in which the elements are arranged in a linear order.
Each list element is an object with an attribute key (data) and one pointer: next.
Last element points to NIL. Head points to first element.



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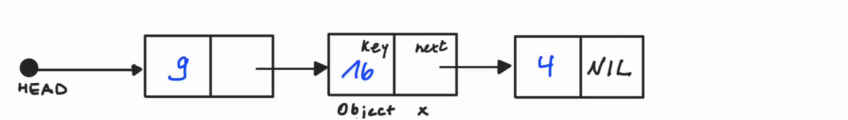

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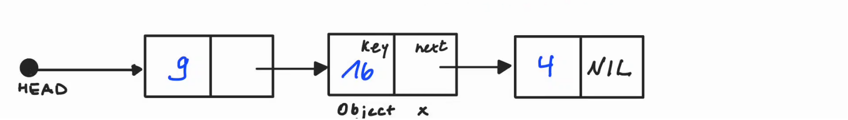

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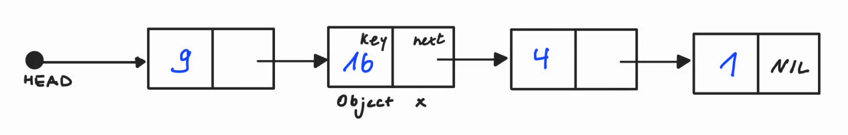

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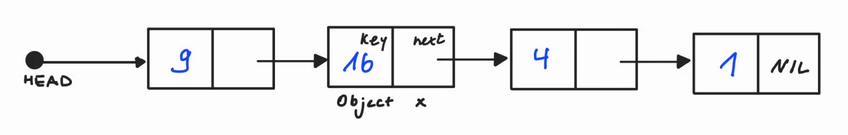

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Such linked lists can be used to realize "dynamic sets" (here the set $\{1, 4, 9, 16\}$)

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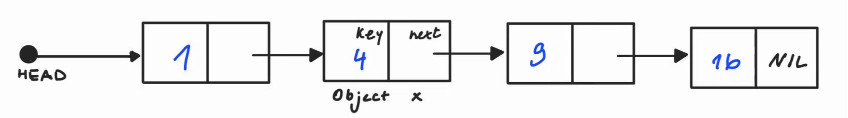

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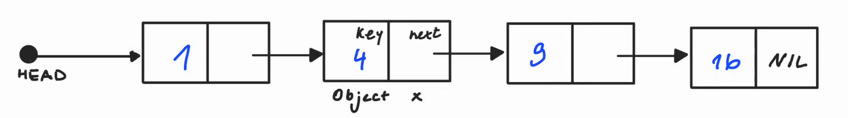

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$\Longrightarrow$ $\Theta(|A|)$ time

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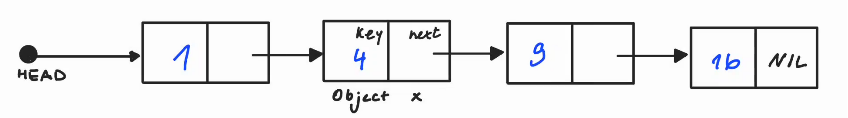

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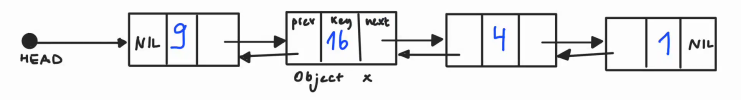

Advantages:

New elements can be placed anywhere in memory and added in constant time before or after a given element by changing the pointers.

Disadvantages:

When searching for an element, you have to go through the list from the first (or last) element to the respective position.
Searching in (sorted) lists L only takes $O(|L|)$ time.

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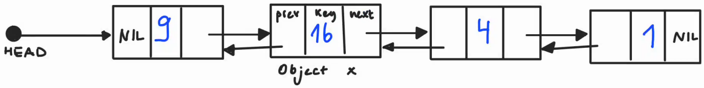

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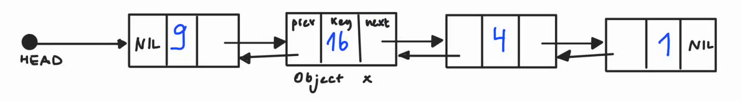

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- **Stacks** follow LIFO = last-in, first-out
  - S.push(x): Inserts item x at the top (last item) of stack S.
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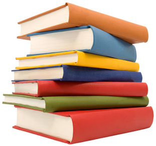

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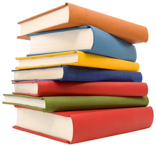

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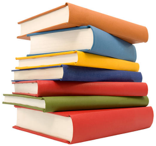


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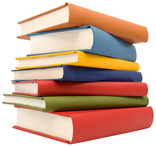

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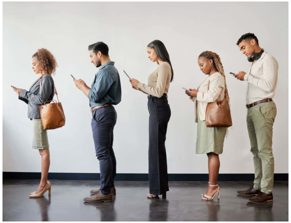

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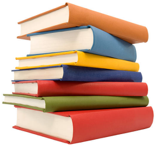

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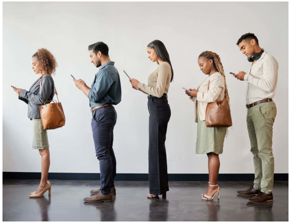

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Part 1-4: Elementary Data Structures (Trees)

Trees form a more general framework than linked list and are defined as "special graphs".

Let us start with the formal definition first.

The next slide contains a lot of definitions that we also need later on (e.g. for heaps, binary search trees, AVL trees, . . .). Most of these defs refer Sec B4 and B5 in the Cormen et al. course-book.

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Part 1-4: Elementary Data Structures (Trees) - some graph theory

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3. *G is connected, and $|E| = |V| - 1$.*
4. *G is acyclic, and $|E| = |V| - 1$.*

A tree $T = (V, E)$ is rooted if there is a distinguished vertex $\rho \in V$, called the root of T

For a rooted tree we can define a partial order $\preceq_T$ on V such $x \preceq_T y$ if y lies on the unique path from the root ρ to x.

If $x \preceq_T y$, then y is an ancestor of x and x a descendant of y

If $x \preceq_T y$ and $\{x, y\} \in E$, then x is child of y and y a parent of x.

A vertex in T without any children is a leaf. A vertex that has a child is an internal or inner vertex.

2 vertices with the same parent are siblings.

A rooted tree is ordered if for every vertex v its children are ordered.

Part 1-4: Elementary Data Structures (Trees) - some graph theory

[for all we give examples - board!!!]

A graph $G = (V, E)$ is a tupel consisting of a vertex set $V := V(G)$ and an edge set $E(G) := E$ that is a subset of the 2-elementary subsets of V.

A path (of length k) is a sequence $P = (v_0, v_1, \ldots, v_k)$ of vertices such that $\{v_i, v_{i+1}\} \in E$, $0 \leq i < k$. $P = (v_0, v_1, \ldots, v_k)$ is also called $v_0 v_k$-path and said to connect v_0 and v_k.

A path P is simple if the vertices $v_0, v_1, \ldots, v_k$ are pairwise distinct. Note that $P = (v_0)$ is a simple path of length 0.

A simple cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ of length $k+1$ is defined by:

$P = (v_0, v_1, \ldots, v_k)$ is a simple path of length $k \geq 2$ and $\{v_k, v_0\} \in E$.

A graph G is connected if for any two vertices $x, y \in V(G)$ there is an xy-path.

Graphs without simple cycles are called acyclic or forest.

A connected acyclic graph is a tree.

Theorem. *The following statements are equivalent for every graph $G = (V, E)$ (exercise):*

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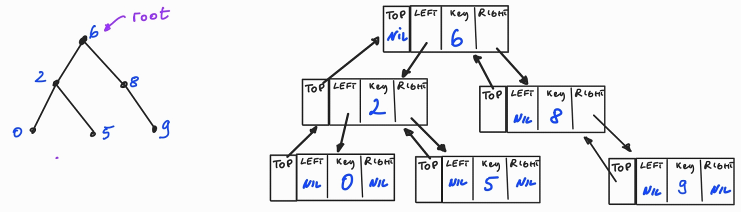
A rooted tree T is **binary** if each vertex as *at most* two children. If T is ordered and binary, then there is a clear distinction between right and left child (even if a vertex has only child).



Advantages:

- New elements can be placed anywhere in memory and added in constant time before or after a given element by changing the pointers.
- Searching in a sorted tree takes $O(h)$ time, with $h =$ height of tree (=longest simple path from root to some leaf).
- In so-called "balanced trees" $h \in O(logn)$ where $n =$ number of vertex (key/data) stored in T [details in upcoming lectures]

Disadvantages:

- Searching in "non-balanced" tree $O(|n|)$ time (as in linked-lists)
- Making a non-balanced tree to a balanced one gets tricky (in particular, insertion of elements is more complicated)

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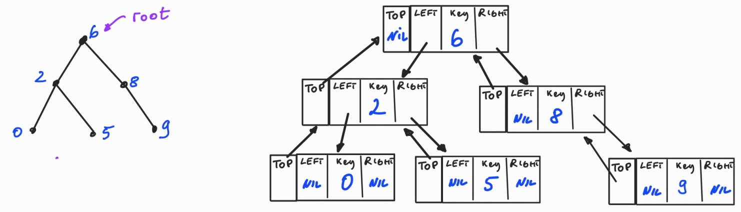

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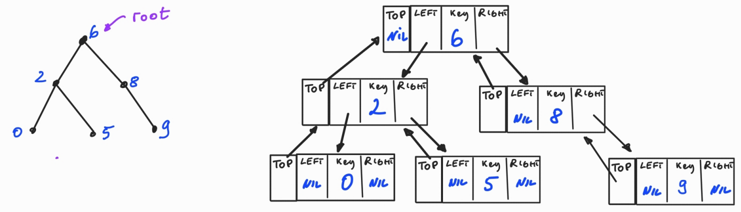

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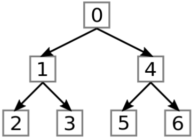
Part 1-4: Elementary Data Structures (Rooted Binary Trees)

Traversal of trees (more details in upcoming lectures).

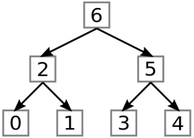
Preorder:

1. visit current vertex
2. recursively traverse left subtree
3. recursively traverse right subtree



Postorder:

1. recursively traverse left subtree
2. recursively traverse right subtree
3. visit current vertex



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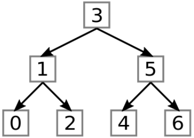

numbers in squares =
order in which nodes are visited

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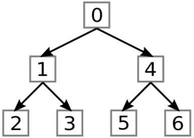

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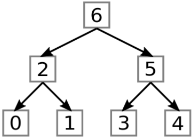

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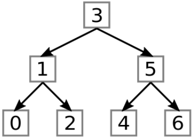

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Part 1-4: Elementary Data Structures

Plenty of other data structures exist and we will examine some of them later in the course