Lecture 11

Last time we saw that being a sheaf is a necessary condi^n to be rep.

? What can we do if we start with a construc- that is not a sheaf?

Thm-Def: Let $\mathcal{C} = k\text{-Aff}$ and $J \in \{fppf, \text{étale}, \text{Zariski}\}$. To each presheaf $\mathcal{F}$ on $\mathcal{C}$ (or k-functor) corresponds a unique J-sheaf $\bar{\mathcal{F}}$ together with a natural transforma^n $\alpha : \mathcal{F} \to \bar{\mathcal{F}}$ s.t.:

$\forall$ J-sheaf $\mathcal{G}$
$\forall$ natural transf. $\eta : \mathcal{F} \to \mathcal{G}$, $\exists!\ \bar{\eta} : \bar{\mathcal{F}} \to \mathcal{G}$
making the following diag commute

$$\begin{tikzcd} \bar{\mathcal{F}} \arrow[r, dashed, "\bar{\eta}"] & \mathcal{G} \\ \mathcal{F} \arrow[u, leftarrow, "\alpha"'] \arrow[ur, "\eta"'] & \end{tikzcd}$$

such a pair $(\bar{\mathcal{F}}, \alpha)$ is called the sheafification of $\mathcal{F}$ wrt J associated to $\mathcal{F}$ for the J-sheaf

– see Görtz/Wedhorn – Algebraic geometry I Schemes with Examples & exercises. Prop 2.24

"Idea of proof" Let $\mathcal{F}$ be a presheaf we def a presheaf $\mathcal{F}'$ as follows: $\forall U \in \mathcal{C}$, $\mathcal{F}'(U) = \varinjlim_{\substack{(U_i \to U) \in Cov(U) \\ i \in I}} \left(\prod \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \right)$

and repeat the process once.

Reminder on direct limits – Let $(I, \leq)$ be a directed set (namely a preordered set in which every finite subset has an upper bound.

A directed system over I is a family $\{A_i : i \in I\}$ of objects indexed by I together with morphisms $f_{ij} : A_i \to A_j$ $\forall\, i \leq j$ such that f_{ii} is the identity on A_i and $\forall\, i \leq j \leq k$ we have $f_{ik} = f_{jk} \circ f_{ij}$.

Let $\mathcal{C}$ be a category and $\langle A_i, f_{ij} \rangle_{i \in I}$ a direct system of objects and morphisms in $\mathcal{C}$.
The direct limit of the direct system $\langle A_i, f_{ij} \rangle$ is a pair $(\varinjlim A_i, \varphi_i) \in Ob\,\mathcal{C} \times (Mor\,\mathcal{C})^I$ s.t. $\varphi_i : A_i \to \varinjlim A_i$ $\forall i \in I$ and $\varphi_i = \varphi_j \circ f_{ij}$ whenever $i \leq j$, that satisfies the following universal property
$\forall (Y, \psi_i) \in Ob\,\mathcal{C} \times (Mor(\mathcal{C}))^I$ such that $\psi_i : A_i \to Y$ and $\psi_i = \psi_j \circ f_{ij}$ $\forall j \leq i$,
$\exists!\ u : \varinjlim A_i \to Y$ such that the following diag commute

$$\begin{tikzcd} A_i \arrow[rr, "f_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddr, bend right, "\psi_i"'] & & A_j \arrow[dl, "\varphi_j"'] \arrow[ddl, bend left, "\psi_j"] \\ & \varinjlim A_i \arrow[d, dashed, "u"] & \\ & Y & \end{tikzcd}$$

In the case we are interested in:

If $\mathcal{U} \in \mathcal{C}$, let $\Sigma_{\mathcal{U}}$ the set of open coverings $\mathcal{V} = \{\mathcal{V}_i\}_{i \in I}$ where $\mathcal{V}_i \neq \mathcal{V}_j$ if $i \neq j$ together with the order given by refinements

$$F'(\mathcal{U}) = \varinjlim_{\mathcal{V} \in \Sigma_{\mathcal{U}}} D(\mathcal{V})$$

$$D(\mathcal{V}) = \{(s_i) \in \prod_{i \in I} F(\mathcal{V}_i) \mid s_i|_{\mathcal{V}_{ij}} = s_j|_{\mathcal{V}_{ij}} \in F(\mathcal{V}_i \times_{\mathcal{U}} \mathcal{V}_j)\}$$

This is a directed system

image of $\mathcal{V}_i \times_{\mathcal{U}} \mathcal{V}_j$ in $\mathcal{V}_i$, resp $\mathcal{V}_j$

We are considering direct limits here that may cause set-theoretical problems, this can be fixed by choosing a universe BUT the limit may not exist in that universe: this happens already for the fpqc sheaves, namely there exist fpqc presheaves that are not fpqc sheaves.

This problem doesn't occur in the case of Zariski, étale and fppf topologies because of the finiteness conditions involved in the def of these topologies. In other words associated sheaves always exist and are independent of any choice of universe.

Remark: Taking twice the limit?

The first limit allows one to get an injective morphism

$$F'(\mathcal{U}) \longrightarrow \prod_i F'(\mathcal{U}_i)$$

but one needs to take the limit again to ensure that the image lands in the kernel of the corresponding double arrow

Exercise: Show that the sheafification of a sheaf is the sheaf itself.

Sheafifica° is the tool we needed to define quotients by group schemes. DGG III §2 & §3 in part. 3.2

Let G be an affine group scheme G acting freely on an affine scheme X on the left.

$\forall R \in k\text{-}Alg$ $G(R) \curvearrowright X(R)$ and we can consider the orbit space $X(R)/G(R)$ for this induced action on R-points. Hence the existence of a k-fun

$$Q: kAlg \longrightarrow Set$$
$$R \longmapsto X(R)/G(R)$$

we are going to sheafify (for the fppf topology as fpqc sheafifica° is not always well defined).

Def The fppf quotient sheaf of X by G, denoted X/G is the fppf sheafification of the above presheaf Q

? Local-global discussion

As mentioned coverings for topologies on $k\text{-}Aff$ can be described in terms of $k\text{-}Alg$, by considering a collec° of algebra homomorphisms $R \longrightarrow R_i$ satisfying the corresponding properties.

Def: Let $\mathcal{J}$ be a Grothendieck topology on $k\text{-}Aff$. A k-functor X satisfies a property $\mathcal{P}$ $\mathcal{J}$-locally if $\exists$ a $\mathcal{J}$-covering $(Spec(k_i))_{i \in I}$ of $Spec(k)$ s.t. X_{k_i} satisfies $\mathcal{P}$ for each $i \in I$.

Remarks: ① In part two k-functors X & Y are $\mathcal{J}$-locally isomorphic if there is a $\mathcal{J}$-covering $(Spec(k_i))_{i\in I}$ of k with $X_{k_i} \cong Y_{k_i}$ $\forall i$.

② A k-functor X satisfies a property $\mathcal{P}$ globally if $\mathcal{P}$ is satisfied locally with respect to the covering of $Spec\, k$ by itself

Example: If k is a field, let $\mathcal{U}, \mathcal{V}$ be two finite dimensional k-vector spaces. Then $W_{\mathcal{U}}$ (def $\forall R \in kAlg$ by $W_{\mathcal{U}}(R) = R \otimes_k \mathcal{U}$) and $W_{\mathcal{V}}$ are fppf-locally isomorphic iff they are globally iso.

In particular the property of being of some given dimension holds globally for a vector space of ⓟ and only if it holds fppf locally.

This property is thus said to be invariant under fppf-descent.

? Which prop of k-functors are invariant under (fppf)-descent?

Namely given a (fppf)-covering $\mathrm{Spec}(k_i)$ of $\mathrm{Spec}(k)$ and a k-scheme X, under which condi° $X \to \mathrm{Spec}(k)$ satisfies P iff $X_{k_i} \to \mathrm{Spec}(k_i)$ $\forall i \in I$?

Thm: Let X, Y be two affine k-schemes and $f: X \to Y$ a morphism. Let k' be a faithfully flat k-algebra. Then for each of the following property (the list is far from being exhaustive), f has this property iff $f_{k'}: X_{k'} \to Y_{k'}$ has this property:

- an iso
- a mono
- an open immersion
- a closed immersion
- flat
- faithfully flat
- smooth
- étale

In part a k-scheme X is flat, ..., iff so is $X_{k'}$

Torsors: **?** Given a k-functor X how can we describe/construct those k-functors X' that are J-locally iso to X?

Let G be a k-gp functor that is wlog an fppf sheaf. This in particular applies to affine group schemes.

If E and E' are k-functors acted on by G, a G-homomorphism (resp isomorphism) that is $E \to E'$ is a natural transforma° that is G-equiv in the obvious sense.

Def: A (right) G-torsor is a k-functor E that is an fppf-sheaf endowed with a right G-ac° s.t.

T1) The induced hom. $E \times G \to E \times E$ def $\forall R \in k\text{-Alg}$ by

$$E(R) \times G(R) \to E(R) \times E(R)$$
$$(e, g) \mapsto (e, e.g)$$

is an isomorphism

T2) $\exists$ a faithfully flat k-algebra k' s.t

$E_{k'} \cong G_{k'}$ as $G_{k'}$-functors

↳ seen as a $G_{k'}$-torsor under right multiplica°

Remark: T1) means that the ac° of $G(R)$ on $E(R)$ is free and transitive $\forall R \in k\text{-Alg}$ unless $E(R)$ is empty (⚠)

Example / Remarks 1. bis.

- If G is representable then any G-torsor is representable (follows from descent theory).
- A G-torsor E is isomorphic to a trivial torsor if and only if $E(k) \neq \phi$ i.e. the structure morphism $E \longrightarrow \mathrm{Spec}(k)$ admits at least a sec$^{\underline{\circ}}$

Indeed: * if $\exists$ such a sec$^{\underline{\circ}}$ $s : \mathrm{Spec}\, k \to E$ then
$$\mathrm{Spec}(k) \times G \longrightarrow E \quad \text{is an iso.}$$
$$(x, g) \longmapsto s(x)g$$

* On the other hand, if E is iso to a trivial G-torsor then $E \simeq \mathrm{Spec}(k) \times G$, the identity element $1_G \in G$ gives the expected sec$^{\underline{\circ}}$ $s = id_x \times 1_G$

T2) means that a torsor is fppf locally trivial.

Ex: ① Consider the ac$^{\underline{\circ}}$ of G on itself by right mult. This makes G into a G-torsor, called the trivial G-torsor.

Clearly G is G-iso to itself $\Longleftrightarrow$ $G(R) \neq \phi$ $\forall R \in k\text{-}Alg$ (it contains the id element)

② (Exercise): let E, F be two k-functors that are also fppf-sheaves. Remember that we have defined:

1) $Hom(E, F)$ as the k-functor s.t $\forall R \in k\text{-}Alg$
$$Hom(E,F)(R) = Hom(E_R, F_R)$$
(that is, $Hom(E,F)$ is the set of all natural transforma$^{\underline{\circ}}$ from E to F).

2) $Isom(E,F)$ as the k-subfunctor of $Hom(E,F)$, s.t $\forall R \in k\text{-}Alg$
$$Isom(E,F)(R) = Isom(E_R, F_R)$$
(that is, $Isom(E,F)$ is the set of all invertible natural transformations from E to F).

In particular $Aut(E) := Isom(E,E)$.

Then $Hom(E,F)$ and $Isom(E,F)$ are fppf-sheaves, $G = Aut(E)$ is an fppf-sheaf of groups that acts on $Isom(E,F)$ on the right by precomposition and if E is fppf-locally isomorphic to F then, under this action, $Isom(E,F)$ is a G-torsor.

Remember that (T1) means that $\forall R \in k\text{-Alg}$ the action of $G(R)$ on $E(R)$ is free and transitive unless $E(R)$ is empty, namely, in our example $\forall R \in k\text{-Alg}$ either there are no isomorphisms between E_R and F_R or there is at least one such iso φ. Then every other iso ψ is obtained by precomposing with an automorphism of E, namely if $\varphi, \psi \in Isom(E,F)$ then $\exists \rho \in Aut(E)$ s.t. $\psi = \varphi\rho$.

? $G(R) \circlearrowright E(R)$ is free and transitive unless $E(R) = \phi$?

Do we have a "control" on this?

Exercise: Let E be an fppf-torsor and assume that there is an $R \in k\text{-Alg}$ s.t. $E(R) \neq \phi$ then $E(S) \neq \phi$ $\forall S \in R\text{-Alg}$.

(3) With the above notations, given a G-torsor T, there is a way to construct the twist of E by T denoted $F := T \wedge_G E$, which is an fppf sheaf such that F is fppf locally isomorphic to E &

In particular, given a scheme $S = \mathrm{Spec}(k)$ ← any ring

(or more generally any scheme)

there is a bijection between vector bundles over S

= locally free sheaves

and Gl_n-torsors, where $n = rk(V) \in \mathbb{N}$

$$\begin{array}{ccc} \text{Vector bundles} & \longrightarrow & Gl_n\text{-torsors} \\ V & \longmapsto & \mathrm{Isom}(V, \mathcal{O}_S^{\oplus n}) \end{array}$$

$\wr$

it is a $\mathrm{Isom}(\mathcal{O}_S^{\oplus n}, \mathcal{O}_S^{\oplus n})$

$\cong$

$Gl_{n,S}$-torsor

$T \cong \mathrm{Isom}(E, F)$.

Hence the fppf G-torsors is equivalent to classifying those F that are fppf-locally isomorphic to E.

We denote

$H^1_{fppf}(k, G)$ = set of all isomorphism classes of G-torsors

first fppf cohomology of G

⚠ In general this problem is as hard as classifying the torsors unless there is a clear reason why the cohomology must be trivial:

Example: Hilbert 90th theorem

The above example on the dimension of f d k-vector spaces seen as vector gps can be formulated as follows: if k is a field then $H^1_{fppf}(k, Gl_n)$ is trivial $\forall n$.

Fibered categories, stacks & descent data:

(?) Given a $\mathcal{T}$-covering k_i of k, and, $\forall i$, a k_i-functor X_i, what conditions do the X_i need to satisfy in order to descend to a global object X, i.e. when does there exist a k-functor X with $X_{k_i} \cong X_i$? This will require:

Def: (Fibered categories) A fibered category $\mathcal{F}$ over a category $\mathcal{C}$ consists of the following data:

(i) $\forall \mathcal{U} \in \mathrm{Ob}(\mathcal{C})$, a category $\mathcal{F}(\mathcal{U})$

(ii) $\forall f: \mathcal{U} \to \mathcal{V}$ in $\mathrm{Mor}(\mathcal{C})$, a functor $f^*: \mathcal{F}(\mathcal{V}) \to \mathcal{F}(\mathcal{U})$

(iii) an isomorphism of functors

$$(\mathrm{Id}\,\mathcal{U})^* \xrightarrow{\sim} \mathrm{Id}_{\mathcal{F}(\mathcal{U})} \quad \forall \mathcal{U}$$

(iv) an isomorphism of functors

$$f^* g^* \longrightarrow (gf)^* \quad \forall\ \mathcal{U} \xrightarrow{f} \mathcal{V} \xrightarrow{g} \mathcal{W}.$$

satisfying the expected compatibility rela$^=$.

Example: $\mathcal{C} = (k\text{-Alg})^\circ$ (the opposite category), $\mathcal{F}(R) = R\text{-Mod}$

$\forall R \in k\text{-Alg}$, and $f: R \to S \in \mathrm{Mor}(\mathcal{C})$

$f^* = - \otimes_S R$ (remember that f correspond to a usual alg hom $S \to R$ as we work in the opposite category).

Then iso in (iii) & (iv) are nothing but the corresp. iso from linear algebra.

Back to the question we are interested in... Once again, thinking with topological spaces is instructive:

Let $\mathcal{U}$ be a topological space and an open covering $(\mathcal{U}_i)_i$ of $\mathcal{U}$. We are given some collec$^=$ of objects X_i one over each $\mathcal{U}_i$, so that we have a map $g_i: X_i \to \mathcal{U}_i$

To be able to glue this to some $g: X \to \mathcal{U}$ we need compatible iso on intersec$^=$, namely

$$\Phi_{ij}: g_j^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \longrightarrow g_i^{-1}(\mathcal{U}_i \cap \mathcal{U}_j).$$

s.t. on triple intersec$^{\circ}$
$U_i \cap U_j \cap U_k$, $\Phi_{ik} = \Phi_{ij} \circ \Phi_{jk}$

Let's formalise this.

Let $\mathcal{C}$ be a site, $U \in \mathrm{Ob}(\mathcal{C})$ and a covering $\mathcal{U} : (U_i \xrightarrow{f_i} U)_{i \in I}$

"Intersec$^{\circ}$ are fibered products":

a fibered product $U_i \times_U U_j$ comes with the two canonical proj $p_i : U_i \times_U U_j \longrightarrow U_i$ & $p_j : U_i \times_U U_j \longrightarrow U_j$
and for a fibered product
$U_i \times_U U_j \times_U U_k$ we write p_{ij}, p_{ik}, p_{jk} for the 3 proj to $U_i \times_U U_j$, ...

If now $\mathcal{F}$ is a fibered category over $\mathcal{C}$ and $X_i \in \mathcal{F}(U_i)$ for each i, we want to glue these into $X \in \mathcal{F}(U)$:

Def. (Descent Data): A descent datum on the family $(X_i)_{i \in I}$ is a set of isomorphisms
$$\Phi_{ij} : p_j^*(X_j) \longrightarrow p_i^*(X_i)$$
in the category $\mathcal{F}(U_i \times_U U_j)$ s.t. $\forall (i,j,k)$ the following cocycle condi$^{\circ}$ is satisfied in $\mathcal{F}(U_i \times_U U_j \times_U U_k)$
$$p_{ik}^*(\Phi_{ik}) = p_{ij}^*(\Phi_{ij})\, p_{jk}^*(\Phi_{jk})$$

The family $(X_i)_{i \in I}$ with a descent datum is called an object with a descent datum.
Morphisms between objects with descent datum are def in the obvious way so that we get a category $\mathcal{F}(\mathcal{U}/_U)$ of objects with descent data.

Example: Stacks.

- To each $X \in \mathcal{F}(U)$ one can associate an object with a descent datum:
set $X_i = f_i^*(X)$ where $f_i : U_i \longrightarrow U$ is the map from the covering & $\Phi_{ij} = p_j^*(f_j^*(X_j)) \longrightarrow p_i^*(f_i^*(X))$
Thus defines a functor $\mathcal{F}(U) \longrightarrow \mathcal{F}(\mathcal{U}/_U)$
If this functor is an equiv of categories $\forall U \in \mathcal{C}$ & $\forall \mathcal{U} \in \mathrm{Co}(U)$, we say that $\mathcal{F}$ is a stack
- $\mathcal{C} : (k\text{-Alg})^{o}$ endowed with fpqc and $\mathcal{F}(R) = R\text{-Mod}$ $\forall R \in (k\text{-Alg})^{o}$ as above. This is a stack over $\mathcal{C}$.

A key step in the proof is to show that the functor $\mathcal{F}(U) \longrightarrow \mathcal{F}(\mathcal{U}/_U)$ is fully faithful when $\mathcal{U}$ is a singleton covering.
Working through the technicality this amounts to showing: if A is a ring & B a faithfully flat A-alg then $\forall M, N \in A\text{-mod}$, the induced diag is exact
$$\mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B) \rightrightarrows \mathrm{Hom}_{B \otimes_A B}(M \otimes_A B \otimes_A B, N \otimes_A \ldots)$$

This allows one to show that rep functors are fpqc-sheaves. (15)

⚠ Faithfully flat descent CANNOT be applied to reduce classifica: pb over general rings to pb over fields. This is because in general the field of frac: of a ring R is not faithfully flat over R. However under mild regularity assump: such reduc: can be made:

Thm: Let X, Y be 2 affine k-schemes of finite presenta: & assume that X is flat. Then a morphism $f : X \to Y$ is an iso iff so is $f_F : X_F \to Y_F$ $\forall F \in k\text{-Alg}$.

Generalities on reductive group schemes.

General philosophy: "Everything" comes from the fibers! Then use descent to derive a general theory.

Def. Let S be a (affine) scheme. A reductive S-gp is a smooth S-affine group scheme $G \to S$ such that the geometric fibers $G_{\bar{s}}$ are connected reductive groups.

A semisimple S-group is a reductive S-group whose geometric fibres are semisimple.

Remark: Actually, it suffices to check reductivity (resp semisimplicity) for a single geometric point over each $s \in S$.

Indeed for any H (linear alg group) over an alg closed field k and any alg closed extension K/k the inclusions

$R_u(H)_K \subset R_u(H_K)$ and $R(H)_K \subset R(H_K)$

are equalities

Exercise!

Remark: In the classical theory, tori are minimal example of reductive groups.

Tori govern an important part of the classical theory for reductive groups.

Diagonalizable groups. Groups of multiplicative type and tori:

Def: Diagonalizable groups.

Let: M be an abelian gp

$k[M]$ be its group algebra $= \bigoplus_{m \in M} k e_m$

= the associative algebra over k whose underlying k-module is the free module over k with multiplica: given on basis elements by the group operation, $e_{m+n} = e_m + e_n$ $\forall m, n \in M$.

$k[M]$ is endowed with a Hopf structure as follows:

$\forall m \in M$, $m(m) = m \otimes m$

$\varepsilon(m) = 1$ and $i(m) = m^{-1}$

and extend by linearity.

An affine group scheme is diagonalizable if it is isomorphic to $\mathrm{Hom}_{k[M]} =: \mathcal{D}(M)$ for some abelian group M.

? What do such groups look like?

→ the classification of finitely generated abelian groups steps in to give the following:

Proposition: If $G = \mathrm{Hom}_A$ is diagonalisable and A is finitely generated as a k-algebra then G is a product of finitely many copies of $\mathbb{G}_m$ and μ_n for various n.

Proof. Let $A = k[M]$. The proof breaks down in a few steps:

① A finitely generated ⟹ M is finitely generated

② If $M = \mathbb{Z}$, then $G = \mathbb{G}_m$ (to show it, identify the Hopf algebras).

③ If $M = \mathbb{Z}/n\mathbb{Z}$, then $G = \mu_n$ (same reasoning as in ②).

④ Conclude, using → the classifica: of f.g ab. gps

→ $k[M \oplus N] \cong k[M] \otimes k[N]$

→ the Yoneda functor sends tensor products to direct products.

Def: A finitely presented affine gp scheme G/k is of mult type if there exists an fppf cover $\{S_i\}$ of $S = \mathrm{Spec}\, k$, s.t. $G \times_k S_i$ is a diagonalizable S_i-gp scheme

- If G_{S_i} is isomorphic to some $\mathcal{D}(\mathbb{Z}^n)_{S_i}$ for each i, we say that G is a torus.

Remarks.

① By fppf descent, G: S-gp scheme of mult. type is faithfully flat and finitely presented over S

$$\begin{tikzcd} G_{S_i} \arrow[r] \arrow[d, "\text{fppf}"'] & G \arrow[d] \\ S_i \arrow[r, "\text{fppf}"'] & S \end{tikzcd}$$

② When $G \simeq D_S(M)$ for some M, one says that G is split (or diagonalisable).

Note: split groups of multiplicative type exist!

Lemma: Let k be a field. A k-gp H of multiplicative type splits after base change to a finite separable extension of k.

Idea of proof: • Enough to show: H_{k_s} is split, where k_s = sep. closure of k. (see Szamuely Galois Groups & Fundamental gps. §5.4)

• Assume $k = k_s$, one aims to show $H \simeq D_k(M)$ for a finitely generated ab. gp M.

• The functor $I = \mathrm{Isom}(H, D_k(M))$ is

check it! $\begin{cases} \text{an fppf-sheaf on the category of } k\text{-schemes} \\ \Rightarrow \text{it is an } \mathrm{Aut}(M)\text{-torsor} \end{cases}$

Descent $\Rightarrow$ techniques I is constant.

③ An equivalent def of tori (in the spirit of the def of being reductive or semi simple):

An S-torus is an S-group scheme $T \to S$ of multiplicative type with smooth connected fibers. (check it is indeed equivalent).

Proposition: If $G \to S$ is a gp scheme of multiplicative type then $\exists!$ closed maximal subtorus $T \subset G$.

Remarks: ① maximal means that

a) T contains all closed subgroups of G

b) This is still the case after arbitrary base change.

we will see later that this indeed makes sense.

② subtorus? Remember that, to make a subgp functor into a subgp scheme one needs the induced monomorphism to be a closed immersion

Lemma: Let G be an affine S-group scheme of finite presentation and H an S-group of mult type. Any monomorphism $j: H \hookrightarrow G$ is a closed immersion.

(In other words, working sheaf theoretically with monomorphism is here equiv. to working algebro-geometrically with closed immersion.

⚠ one could remove finite pres assump: (more technical, not needed here)

<u>Idea of proof for the lemma:</u> (Conrad, B.1.3)

1) The situa: is governed by the noetherian case, here use mono + proper (f.t., separated univ closed) $\Rightarrow$ closed imm. & use valuative crit for properness to reduce to R: Henselian (local ring with max ideal m s.t. $\forall P$ monic in $R[x]$ any factoriza: of $\bar{P}$ in $R/m[x]$ into a product of coprime monic can be lifted to a dec in $R[x]$.)

2) Then show that $\exists$ faithfully flat base change $R \to R'$ so that $H_{R'}$ is split

3) Then reduce to $H \cong \mathbb{G}_m$ (technical!) & k: residue field of R is alg closed and use this to show that the transla: ac: of H on $G = \operatorname{Spec} A$ def a $\mathbb{Z}$-grading

$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$

and conclude.

<u>Idea of the proof of the thm:</u>

① Use fppf descent to restrict to the split case $G = D_S(\mathcal{M})$

Let $\mathcal{M}' = \mathcal{M}/\mathcal{M}_{tor}$ be the max torsion free quotient of $\mathcal{M}$ and $T = D_S(\mathcal{M}') \subset D_S(\mathcal{M}) = G$

② Show that every closed subtorus of G is contained in T (working Zariski locally it is enough to show this for those of shape $D_S(\mathcal{M}/N)$, $N \trianglelefteq \mathcal{M}$)

torus by assump: $\Rightarrow \mathcal{M}/N$ torsion free $\Rightarrow \mathcal{M}/N$ is dominated by $\mathcal{M}'$ as a quotient.

<u>Remark:</u>

Defining reductive gp schemes as above is <u>at least</u> reasonable in the sense that reductivity of a fibre is inherited by nearby fibres for any smooth affine gp scheme with connected fibres.

<u>Prop:</u> Let $G \to S$ smooth affine gp scheme and let $H_0 \subset G_s$ be a mult. type subgp of G_s, $s \in S$.

$\exists$ an étale neighborhood (S', s') of (S, s) with $k(s') = k(s)$ & a mult type subgp $H' \subset G_{S'}$ s.t. $H'_{s'} = H_0$ inside $(G_{S'})_{s'} = G_s$