Lecture 9 - 1

Coroots ?

How do we define coroots ?

A starting point: The SL_2 example.

$T = D$ identifies with $\mathbb{G}_m$ via

$$\begin{aligned} \mathbb{G}_m &\longrightarrow T \\ c &\longmapsto \begin{pmatrix} c & \\ & 1/c \end{pmatrix} \end{aligned}$$

roots for the D-ac° are

$$c^2 : T \longrightarrow \mathbb{G}_m, \quad \begin{pmatrix} c & \\ & 1/c \end{pmatrix} \longmapsto c^2 \qquad\qquad c^{-2} : T \longrightarrow \mathbb{G}_m, \quad \begin{pmatrix} c & \\ & 1/c \end{pmatrix} \longmapsto c^{-2}$$

This admits the following generaliza°:

Thm: For each $a \in \Phi(G,T)$ $\exists$ a hom. $\varphi_a : SL_2 \to G$ carrying D into T and $U^{\pm}$ isomorphically onto the root gps $U_{\pm a}$

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Such a φ_a is an isogeny onto $\mathcal{D}(Z_G(T_a))$ with $\ker \varphi_a \subset \mu_2$, it is unique up to $T(k)$-conjuga° in G.

Proof: Conrad Thm 1 2.7.

Idea: Replace (G,T) by $(\mathcal{D}(Z_G(T_a)), T'_a)$

$$\begin{tikzcd} SL_2 \arrow[r, dashed, "\varphi_a"] \arrow[d, "q"'] & G \arrow[dl, "f"] \\ PGL_2 & \end{tikzcd}$$

canonical isogeny

?

semi-simple of rk 1 hence G is iso to either SL_2 or PGL_2

an isogeny complement to T_a given by $T \cap \mathcal{D}(Z_G(T_a^0)_{red})$

f is def as $f : G \longrightarrow \mathrm{Aut}_{\mathbb{P}^1_k / k} = PGL_2$ associated to the left transla° action of G on G/B

Def: The coroot associated to (G,T,a) is the cocharacter

$$a^\vee : \mathbb{G}_m \longrightarrow D \longrightarrow T$$
$$c \longmapsto \varphi_a\begin{pmatrix} c & \\ & 1/c \end{pmatrix}$$

by what preceeds it is unaffected by $T(k)$-conj on G namely it is intrinsic.

Remark: It follows from examples for both SL_2 and PGL_2 that in general $\langle a, a^\vee \rangle = 2$ $\forall$ connected red k-gp G and max torus $T \subset G$.

Note: $a^\vee$ is a param. (with kernel 1 or μ_2) of the 1-dim torus $(T \cap \mathcal{D}(Z_G(T_a)))^\circ_{red} = a^\vee(\mathbb{G}_m)$ that is isogeny complement to T_a in T.

Ex ① $G = Sl_2$ $T = D$

$\varphi_a = id$ and $a^\vee(c) = \mathrm{diag}\begin{pmatrix} c & \\ & 1/c \end{pmatrix}$

$\langle a, a^\vee \rangle = 2 \in \mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$

namely $a(a^\vee(c)) = c^2$

as $\underbrace{\begin{pmatrix} 1 & \\ & 1/c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}_{\mathcal{U}_a}\begin{pmatrix} 1 & \\ & 1/c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c^2 x \\ 0 & 1 \end{pmatrix}$

& the adj ac° of $a^\vee(c)$ on $\mathfrak{g}_a = \mathrm{Lie}(\mathcal{U}_a)$ is scaling by c^2.

② $G = PGl_2$ $T = \bar{D}$ $\mathbb{G}_m \simeq T$ via

$c \mapsto \begin{pmatrix} c & \\ & 1 \end{pmatrix} \bmod \mathbb{G}_m$

$\bar{a} : T \longrightarrow \mathbb{G}_m$ is a root whose root space consists of upper triang nilp matrices in $\mathfrak{pgl}_2$ (same for $-\bar{a}$ with lower).

So $\varphi_{\bar{a}} : Sl_2 \longrightarrow G$ is the canonical proj

- **Def**: A root datum is a 4-tuple $(X, \Phi, X^\vee, \Phi^\vee)$ where

① X and $X^\vee$ are finite free $\mathbb{Z}$-modules such that $\exists$ a perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \longrightarrow \mathbb{Z}$

② $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ are finite subsets, for which $\exists$ a map $\Phi \longrightarrow \Phi^\vee$, $a \longmapsto a^\vee$ such that

RD I $\langle a, a^\vee \rangle = 2$

RD II $\forall a \in \Phi$, $\forall a^\vee \in \Phi^\vee$, the induced reflections

$s_a : X \xrightarrow{\sim} X$

$x \longmapsto s_a(x) = x - \langle x, a^\vee \rangle a$

and $s_{a^\vee} : X^\vee \xrightarrow{\sim} X^\vee$

$\lambda \longmapsto s_{a^\vee}(\lambda) = \lambda - \langle a, \lambda \rangle a^\vee$

satisfy $s_a(\Phi) \subset \Phi$ and $s_{a^\vee}(\Phi^\vee) \subset \Phi^\vee$

- **The root datum** $(X, \Phi, X^\vee, \Phi^\vee)$ is reduced if moreover $\mathbb{Q}a \cap \Phi = \{\pm a\} \subseteq X_{\mathbb{Q}}$ $\forall a \in \Phi$

Remarks: ① RD I is equivalent to any of the following

① $\forall a \in \Phi$, $s_a^2 = id$ ② $\forall a \in \Phi$ $s_a(a) = -a$

(1$^\vee$) $\forall a \in \Phi^\vee$, $s_a^\vee s_a^\vee = id$

(2$^\vee$) $\forall a \in \Phi^\vee$, $s_a^\vee(a^\vee) = -a^\vee$

(2) RD I together with RD II imply that

• $\Phi = -\Phi$ • $\Phi^\vee = -\Phi^\vee$ • $0 \notin \Phi$ • $0 \notin \Phi^\vee$

Lemma: The map $\vee : \Phi \to \Phi^\vee$, $a \mapsto a^\vee$ is bijective. More generally, a coroot $c^\vee$ is uniquely determined by the function $\Phi \to \mathbb{Z}$, $a \mapsto \langle a, c^\vee \rangle$. As a consequence the map $\Phi \to \Phi^\vee$, $a \mapsto a^\vee$ is uniquely determined, that is, if $f : \Phi \to \Phi^\vee$ is a bijection with respect to which the axioms are satisfied then $f(a) = a^\vee$ $\forall a \in \Phi$. Moreover $\forall a, b \in \Phi$, $s_a(b)^\vee = s_a^\vee(b^\vee)$

Proof. Let $a, b \in \Phi$ s.t $\langle x, a^\vee \rangle = \langle x, b^\vee \rangle$ $\forall x \in \Phi$

then $s_b(a) = a - \langle a, b^\vee \rangle b = a - \langle a, a^\vee \rangle b$
$= a - 2b$

and similarly $s_a(b) = b - 2a$

Hence $s_b s_a(a) = s_b(-a) = -a + 2b = a + 2(b-a)$

$s_b s_a(b-a) = s_b(b-a) = b - a$

hence $(s_b s_a)^n(a) = a + 2n(b-a) \in \Phi$ by RD II

But Φ is finite, so $a = b$.

Now, to show that $\vee$ is uniquely determined by the root datum it is enough to show that the functional $\langle \cdot, c^\vee \rangle$ on the $\mathbb{Q}$-vector space $V = X \otimes_{\mathbb{Z}} \mathbb{Q}$ is uniquely determined by the root c above for any possible bijection $\Phi \to \Phi^\vee$, $c \mapsto c^\vee$ that satisfies the root datum axioms.

Let us fix such a bijection and define a reflection of V, $r_c : V \to V$, $x \mapsto x - \langle x, c^\vee \rangle c$ such that $r_c(\Phi) = \Phi$ and $r_c(c) = -c$. This reflection is uniquely determined by c, so r_c is independent of the choice of $c \mapsto c^\vee$

Remark: If $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum then so is the dual root datum $(X^\vee, \Phi^\vee, X, \Phi)$.

Def: Let $\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$ and $\mathcal{R}' = (X', \Phi', X'^\vee, \Phi'^\vee)$ be two root data. Let $f : X' \to X$ be a linear map and ${}^t f : X^\vee \to X'^\vee$ be the transposed morphism. Then $f : \mathcal{R}' \to \mathcal{R}$ is a morphism of root data

if f induces a bijection $\Phi' \to \Phi$ and
${}^t f$ induces a bijection $\Phi^\vee \to \Phi^{\vee\prime}$ (7)

Note that in this case ${}^t f$ is a morphism of dual root data.

def: A morphism of root data $f: X \to X'$ is:

① An isogeny if $f: X' \to X$ is injective of finite cokernel

② An isomorphism if so is $f: X' \to X$

SUMMARY: To a connected reductive k-group G together with a max torus $T \subseteq G$ we have associated

① The lattices of characters $X(T)$ and cocharacters $X_*(T)$,

② Two finite subsets $\Phi \subset X(T) \setminus \{0\}$ and $\Phi^\vee \subset X_*(T) \setminus \{0\}$ as well as a bijection $\Phi \to \Phi^\vee$, $a \mapsto a^\vee$

s.t $\forall a \in \Phi$ $\langle a, a^\vee \rangle = 2$ (? To get a root datum, one needs to check that $\forall a \in \Phi$ the linear end

$s_a: X \to X$, $x \mapsto x - \langle x, a^\vee \rangle a$ and $s_{a^\vee}: X^\vee \to X^\vee$, $\lambda \mapsto \lambda - \langle a, \lambda \rangle a^\vee$

are s.t $s_a(\Phi) \subseteq \Phi$ and $s_{a^\vee}(\Phi^\vee) \subseteq \Phi^\vee$

This is Prop 1.3.2 of Conrad's notes: (8)

Idea: Denote by T_a = max codim 1 torus in T killed by $a \in \Phi$

$= \ker(a)^\circ_{red}$

$G_a = \mathcal{D}(Z_G(T_a)) = \langle U_a, U_{-a} \rangle$

= closed subgp admitting PGL_2 as an isogeneous quotient

So that $T = T'_a \cdot T$ almost direct product, namely $T'_a \cdot T \to G$ is surj with finite kernel.

$T'_a = (T \cap G_a)^\circ_{red}$

$= a^\vee(\mathbb{G}_m)$

The Weyl group $W_{G_a}(T'_a) := N_{G_a}(T'_a) / Z_{G_a}(T'_a) = N_{G_a}(T'_a) / T'_a$ has order 2

has a unique non trivial element of order 2. Choose n_a be one of its representative in $N_{G_a}(k)^{T'}$ (note that n_a can be chosen as $\varphi_a\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$) ← The standard Weyl element of SL_2

This is ensured by the Thm defining the coroots

Then n_a acts on T as the identity on T_a and

as the inversion on T'_a.
One can show that in part u_a acts on $X(T)$ as the reflection s_a. But u_a, as any element of $N_{G(k)}(T)$ preserves Φ, hence the result for s_a.

$N_{G_a(k)}(T'_a)$

note that $g \in N_{G_a(k)}(T'_a) \subseteq \mathcal{D}(Z_G(T_a))$ hence if g normalizes T'_a it normalizes T.

Then use that $s_a^\vee = s_{a^\vee}$ to conclude that $s_{a^\vee}(\Phi^\vee) = \Phi^\vee$.

Def: Let G be a connected red k-group and let $T \subseteq G$ be a maximal torus. The 4-tuple $(X(T), \Phi(G,T), X_*(T), \Phi(G,T)^\vee) =: \mathcal{R}(G,T)$ is the $\underbrace{\text{(reduced) root datum}}$ associated to (G,T)

non reduceness may occur for instance when k is not perfect (so in the settings of this current lecture this never happens).

Theorem: The reduced root datum $\mathcal{R}(G,T)$ associated to a connected red k-group G and max torus $T \subseteq G$ determines (G,T) uniquely up to isomorphism.
Namely, for any 2 pairs (G,T) and (G',T') every isomorphism $\mathcal{R}(G,T) \simeq \mathcal{R}(G',T')$ arises from an isomorphism $(G,T) \simeq (G',T')$ that is unique up to $T(k)$ & $T'(k)$ conjugacy, and every reduced root datum is isomorphic to $\mathcal{R}(G,T)$ for some pair (G,T) over k.

Remarks: ① The set of roots Φ spans $X(T)$ over $\mathbb{Z}$ if and only if $Z(G) = \{1\}$

② $\Phi(G,T)$ is empty iff G is a torus (or equiv. G is solvable)

③ A reduced root datum $\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$ is semi-simple if Φ spans $X_{\mathbb{Q}}$ over $\mathbb{Q}$.
A semisimple root datum is adjoint if $\mathbb{Z}\Phi = X$ and is simply connected if $\mathbb{Z}\Phi^\vee = X^\vee$

? Root datum vs root system? To classify semi simple complex Lie alg one needs a coarser no. than that of a root datum.

Def: A root system is a pair (V, Φ) where V is a finite $\mathbb{Q}$-vs and Φ is a finite spanning set $\Phi \subset V \setminus \{0\}$ s.t. $\forall a \in \Phi$, $\exists$ a reflection

$s_a : v \longmapsto v - \lambda(v) a$, $\lambda \in V^\vee$ and

$$s_a(\Phi) = \Phi$$
$$s_a(a) = -a$$
$$\lambda(\Phi) \subset \mathbb{Z}$$

such a reflection is unique.

Note that if $(X, \Phi, X^\vee, \Phi^\vee)$ is a root datum then the $\mathbb{Q}$-span V of Φ in $X_{\mathbb{Q}} = X \otimes_{\mathbb{Z}} \mathbb{Q}$ together with Φ is a root system.

Grothendieck topologies, torsors and descent

A first example:

Consider the two real Lie alg

- $\mathfrak{sl}_2(\mathbb{R}) = \{ M \in M_2(\mathbb{R}) \mid tr(M) = 0 \}$ that can be viewed as a subspace of $\mathfrak{sl}_2(\mathbb{C})$ via
  $\mathfrak{sl}_2(\mathbb{R}) = \{ M \in \mathfrak{sl}_2(\mathbb{C}) \mid \overline{M} = M \}$
- $\mathfrak{su}_2(\mathbb{R}) = \{ M \in \mathfrak{sl}_2(\mathbb{C}) \mid \overline{M} = -M^t \}$

These are non isomorphic real Lie alg, but they are "locally" iso in the sense that

$$\mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{su}_2(\mathbb{R}) \otimes \mathbb{C}$$

In the same way $SL_{2,\mathbb{R}}$ and $SU_{2,\mathbb{R}}$ are

$SL_{2,\mathbb{C}}$ cut by the equation $\overline{M} = (M^t)^{-1}$

not isomorphic but become isomorphic after base change to $\mathbb{C}$

? What do I mean when I write "locally"?

From now on k is any (commutative, unital) ring.

Remember that an affine scheme X is a locally ringed space $(X, \mathcal{O}_X)$

$\uparrow$ the structure sheaf of the topological space X

Namely, to any open subset $\mathcal{U} \subseteq X$ and any inclusion map $\mathcal{U} \subset \mathcal{V}$, $\mathcal{O}_X$ associates an object $\mathcal{O}_X(\mathcal{U})$ (here a local ring) and a restric° map $res_{\mathcal{V},\mathcal{U}}$ (here a morphism of local rings) that satisfies the axioms you're expecting (namely $res_{\mathcal{U},\mathcal{U}} = id$, $res_{\mathcal{V}\mathcal{U}} \circ res_{\mathcal{W}\mathcal{V}} = res_{\mathcal{U}\mathcal{W}}$ $\forall \mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{W}$) + others, called the gluing axioms.

These are made to bound together local and global datas namely $\mathcal{O}_X(\mathcal{U})$ and $\mathcal{O}_X(X)$ for $\mathcal{U}$ a "small" open subset.

? This is what we need to formalize $\boxed{\text{and}}$ to formalize within the categorical approach we've followed since the 1st lecture

Roughly speaking we need to:

- Introduce the no° of topology on a category
- This being done we want to be able to cover an object $\mathcal{U}$ by a family $(\mathcal{U}_i)_{i \in I}$ of "open neighborhoods"
- A last hope there would be that to get a property (globally) on $\mathcal{U}$ it will be enough to check it locally (on the $\mathcal{U}_i$'s)

Example: X = topological space

We associate to X the category $\hat{X}$ whose objects are all open subsets of X and whose morphisms are inclusions of open sets. If $\mathcal{U} \subset X$ is open then

① $\mathcal{U}$ is a covering of itself

② If $(\mathcal{U}_i)_{i \in I}$ is a covering of $\mathcal{U}$ and if for each $i \in I$, $(\mathcal{U}_{ij})_{j \in J}$ is a covering of $\mathcal{U}_i$ then $(\mathcal{U}_{ij})_{i \in I, j \in J}$ is a covering of $\mathcal{U}$

③ If $\mathcal{V} \subset \mathcal{U}$ is an open subset and $(\mathcal{U}_i)_{i \in I}$ is a covering of $\mathcal{U}$, then $(\mathcal{U}_i \cap \mathcal{V})_{i \in I}$ is a covering of $\mathcal{V}$.

In categorical terms for $\hat{X}$, a morphism $\mathcal{V} \to \mathcal{U}$ is the inclusion of $\mathcal{V}$ in $\mathcal{U}$. A covering is then a family of morphisms $\mathcal{U}_i \to \mathcal{U}$ in $\hat{X}$. Now, the intersec° of two open subsets $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{U}$ is nothing but the pullback (exercise!)

$$\begin{tikzcd}
\mathcal{V} \times_{\mathcal{U}} \mathcal{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V} \arrow[d] \\
\mathcal{W} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

The categorical analogues of ①...③ then writes

C1 If $\mathcal{V} \to \mathcal{U}$ is an iso then $\mathcal{V} \to \mathcal{U}$ is a covering of $\mathcal{U}$.

C2 Same as ②

C3 For each morphism $\mathcal{V} \to \mathcal{U}$ and each covering $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{U}$, the family $(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{V})_{i\in I}$ is a covering of $\mathcal{V}$.

This starting point is to be generalized to other categories:

If $\mathcal{C}$ is a category and $\mathcal{U} \in \mathrm{Ob}(\mathcal{C})$, a covering of $\mathcal{U}$ is any family of morphisms $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$

Grothendieck topologies are def from coverings that behave well.

Def. Let $\mathcal{C}$ be a category. A Grothendieck topology on $\mathcal{C}$ consists, for each $\mathcal{U} \in \mathcal{C}$ of a collection $\mathrm{cov}(\mathcal{U})$ whose elements are coverings of $\mathcal{U}$ s.t. C1 - C3 hold (where C3 is to be understood that pullbacks exist in $\mathcal{C}$).

- A site is a category $\mathcal{C}$ endowed with a topology J. We write $\mathcal{C}_J$ such a site.

Example (following the preceeding example).

For each topological space X, $\hat{X}_{open}$ is a site where Open is the topology whose coverings are all open coverings.

Grothendieck topologies on the cat of affine schemes

We stick to affine schemes as we've limited ourselves to this setting in the framework of this course. But what follows easily extends to schemes.

(1) The Zariski topology.

Each affine k-scheme $\mathcal{U} = \mathrm{Spec}\,A$ is a topological space under the Zariski topology

We def the Zariski topology on the category of affine k-schemes is given by letting each $\mathrm{cov}(\mathcal{U})$ consist of all open coverings of $\mathcal{U}$ by Zariski open immersions.

From a functor of points point of view, $\mathcal{U} \subseteq \mathrm{Hom}_A$ is open if $\exists$ an ideal $J \subset A$ st $\forall R \in k\text{-Alg}$

$$\mathcal{U}(R) = \left\{ \Phi \in \mathrm{Hom}_A(R) \;\middle|\; R\Phi(J) = R \right\}$$

(which amounts to saying that $\mathcal{U}$ corresponds to the complement of the zero locus of J in $\mathrm{Spec}(A)$).

An open immersion $i : \mathcal{V} \to \mathrm{Hom}_A$ is then a monomorphism s.t the image functor is an open subfunctor of Hom_A

Hence $(\mathcal{U}_i \to \mathcal{U})_i \in \underbrace{\mathrm{cov}(\mathcal{U})}_{\text{for the Zariski top}}$ when $\mathcal{U}_i \to \mathcal{U}$ is an open immersion

and the $\mathcal{U}_i \to \mathcal{U}$'s are jointly surj

namely $\mathcal{U}_i = \mathrm{Hom}_{A_i}$

$\mathcal{U} = \mathrm{Hom}_A$

and $\forall\, \mathfrak{p} \subset A$ prime $\exists\, i \in I$ and $\mathfrak{q} \subset A_i$ s.t $\mathfrak{p} = \varphi_i^{-1}(\mathfrak{q})$

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Small vs big site:

$(k\mathrm{Aff}\cdot\mathrm{sch})_{\mathrm{Zar}}$ def above is referred as the big Zariski site of Affine k-schemes as, as a category, we consider the whole category k-Aff.

The small Zariski site is obtained by restricting ourselves to consider only morphisms compatible with the top.

The terminology big vs small site generalize to any other topology.

⚠ If k is a field and $\mathcal{U} = \mathrm{Hom}_k$ there is no non trivial Zariski covering of $\mathcal{U}$ (as there are no non zero proper ideals) ...

⇝ This calls for stronger topologies!

② The étale topology

def A morphism $X = \mathrm{Hom}_A \longrightarrow S = \mathrm{Hom}_B$ is étale if X is a finitely presented B-scheme and if for each $R \in B\text{-Alg}$ & any $J \triangleleft R$ ideal s.t $J^2 = 0$ the canonical map $X(R) \longrightarrow X(R/J)$ is bij (note that for smoothness it was asked to be surj)

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Ex - Prop: Let F be a field. $A \in F\text{-Alg}$ is étale iff A is a product of finite separable field extensions of F.

As a consequence a field ext. is étale iff it is finite and separable

Fiberwise criterion for étale morphisms.

Let $f: X \longrightarrow S$ making X into a flat B-scheme of finite presentation.

f is étale iff X_F is étale $\forall F \in B\text{-Alg}$ that is a field.

Def: The étale topology of k-Aff is the data of $(\mathcal{U}_i \longrightarrow \mathcal{U})_i \in \mathrm{Cov}(\mathcal{U})$

$\mathcal{U}_i \longrightarrow \mathcal{U}$ étale & the $\mathcal{U}_i \longrightarrow \mathcal{U}$'s are jointly surj

Unfortunately the sep condition is still a bit too strict, hence the introduction of the

fppf topology.