Algebraic groups - Lecture 7

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We fix once and for all an algebraically closed field k

Remember: • A linear algebraic group G is a smooth affine k-group scheme

In particular X is finitely gen
$\Rightarrow G(k)$ is Zariski dense in G.

• linear alg. grps G correspond exactly to smooth k-groups that are finitely generated and for which there exists a closed immersion of k-group schemes $j : G \hookrightarrow GL_n$.

• Closed immersions $j : G \hookrightarrow GL_n$ correspond exactly to faithful representations (i.e $\ker(j)$ is trivial)

• Any k-subgroup functor $H \subseteq G$, where G is linear, is a closed subgroup of G.

• The Jordan dec. exists and is well def $\forall g \in G(k)$ allowing to uniquely define (up to commutativity) a decomposition of g as a product $g_u g_{ss} = g_{ss} g_u$

unipotent i.e $j(g_u)$ is a unip endom.

semi-simple i.e $j(g_{ss})$ is diag

This doesn't depend on j

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An algebraic group G is:

(1) unipotent if $\forall g \in G(k)$, $g = g_u$
$\Leftrightarrow$ G has a composi° series (def on the k-points) whose successive quotients are G_a if $\mathrm{char}(k) = 0$ or G_a, α_p & $\mathbb{Z}/p\mathbb{Z}$ if $\mathrm{char}(k) > 0$
$\Leftrightarrow$ G is isomorphic to a linear alg subgroup of U_n

The alg group of strictly upper triang matrices

(2) solvable if $G(k)$ is solvable as an abstract group (namely its derived series terminates)
$\Leftrightarrow$ G is isomorphic to a linear algebraic subgroup of B_n

alg subgp of upper triang mat.

$\Leftrightarrow$ G has a composi° series whose successive quotients are commutative affine alg k-gps

(3) reductive if $R_u(G) = \{1\}$

max connected unipotent (k)-normal linear alg subgroup of G = unipotent radical

(ex: $G = GL_n$)

(4) semi-simple if $R(G) = \{1\}$

(ex: $G = SL_n$)

max connected solvable (k)-normal linear alg subgroup of G = radical of G

Note that $R_u(G)$ and $R(G)$ are well def if linear alg grp G. Indeed:

If $U, U' \triangleleft G$ are connected normal unip. linear alg subgroups of G then $U.U' \triangleleft G$ is closed and unipotent

It is a quotient of $U \rtimes U'$ def in a scheme theoretic way to be the semi-direct product on R-pts $\forall R_i \in k\,Alg$.

One concludes by dim considera°.

The existence of $R(G)$ is obtained in the same way.

? Compatibility with normal subgroups & quotients?

Prop: ① If $H \triangleleft G$ is a normal linear alg subgrp then $R_u(H) = (H \cap R_u(G))^\circ_{red}$ and $R(H) = (H \cap R(G))^\circ_{red}$

given a group scheme $\mathcal{G}$

- we denote by $\mathcal{G}^\circ$ the unique, closed, connected geometrically irreducible subgroup containing the identity element.

when $k = \bar{k}$ "irreducible = red"

- $\mathcal{G}_{red}$ = is the reduced (= smooth here as k is perfect) subscheme of $\mathcal{G}$, namely it is the unique largest closed subscheme of $\mathcal{G}$ with no non-zero nilp element. In part both underlying topological spaces of $\mathcal{G}$ and $\mathcal{G}_{red}$ are the same.

Note that the formation of the underlying reduced scheme is local for the Zariski topology $\Rightarrow$ taking the identity component and the reduced subscheme commute: $(\mathcal{G}^\circ)_{red} = (\mathcal{G}_{red})^\circ$.

② If $\pi : G \to H$ is a surj homomorphism of linear alg k-groups then

$R_u(H) = \pi(R_u(G))$ and likewise for radicals.

Remark: We indeed have to consider the reduced part of the intersection as the intersection of two smooth schemes need not being smooth

E.g. In $G = \mathbb{G}_a^2$ consider: * the closed k-subgroup H_1 given by the equation $x^p + x = y$, so $H_1 \cong \mathbb{G}_a$ hence is smooth

* the additive subgroup of G defined by the equation $x = y$.

Then the intersection $H = H_1 \cap H_2$ is given by the equations $x = y$ and $x^p = 0$ so is isomorphic to $\alpha_p = \mathrm{Spec}(k[t]/t^p)$ which is not smooth in char $p > 0$

Proof of prop: ① Use that $R_u(H) \triangleleft G$ as it is a characteristic subgrp of G.

② Linear alg gps _ Borel 14.11

It follows directly from the above proposition that semi-simplicity and reductivity are inherited by normal linear alg. subgroups and induced on the image of hom.

Solvable subgroups of a linear algebraic group

<u>Def</u>: (Borel and parabolic subgroups): Let G be a linear algebraic group

- a Borel subgroup of G is a maximal connected solvable linear algebraic subgroup of G.
- a parabolic subgroup of G is a (linear alg) subgp of G that contains a Borel subgroup.

<u>Note</u>: • Classically parabolic subgroups of G are def to be the (linear alg) subgroups $P \subset G$ s.t the quasi-proj quotient $G/_P$ is projective. For the equiv. of def see Borel. Linear Alg groups 11.2 Corollary.

The classical def is motivated by:

<u>Theorem</u>: Let G be a linear alg gp over k. The Borel subgroups of G are all $G(k)$-conjugate to each other and these are precisely the connected solvable linear algebraic subgroups B s.t the quasi-proj quotient

scheme $G/_B$ is projective.

<u>Proof</u>: see Borel. Linear alg. gps - 11.1 & 11.2

• As you can notice Borel subgps are assumed to be smooth (as they occur as linear alg subgps of G).

• For parabolics we can actually consider all subgps (and not only the smooth ones) that either contain B or such that the quotient $G/_P$ is projective. This is still an equivalence. In this course we always require parabolic subgps to be smooth (following SGA3, Conrad's & Gille's notes). For details on the non smooth case see Haccon, Parab subgps in char 2 & 3. Journal of LMS. Note that the "non-smooth" case can only occur in char $p > 0$ as any alg affine k-gp is smooth in char 0 (this is due to Cartier see for instance Waterhouse 11.4 for a proof in english).

Finally let me mention that both smooth and non smooth parab subgps of G over an alg closed field are now classified.

<u>Theorem</u>: The smooth parab subgps P of any connected linear alg gps G are connected

⑦

and $N_{G(k)}(P) = P(k)$

If G is connected and reductive one even has the scheme theoretic equality $N_G(P) = P$

Proof: ① Borel Linear Alg grps 11.16

② Conrad's notes Cor. 5.2.8

Note that this in part implies that the normalizer of a smooth! parab subgrp is smooth.

def in the scheme theoretic way we did in lecture 4.

Examples: • For $G = GL_n \supset B_n$ is a Borel subgp.

• $G = GL_n$ again. Some obvious parab subgps occur as subgps of matrices

$$M = \begin{pmatrix} M_1 & * & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & M_r \end{pmatrix}, \quad 1 \leq r \leq n$$

with $M_i \in GL_{a_i}(k)$, $\sum_{i=1}^{r} a_i = n$.

⑧

In SGA 3 reductive groups are connected by def, while in Borel, Linear alg gps they are not

Conrad's notes follow Borel's convention for red groups def over alg. closed fields.

We will see later that fibral connectedness condition has much more importance as for instance, it ensures that the connected closed component of the identity is indeed

(for a smooth gp scheme $G \longrightarrow S$ (any base scheme) of finite presentation

$\exists!$ open subgroup scheme $G^\circ \subset G$ s.t $(G^\circ)_s$ is the connected comp. of id $\forall s \in S$

Tori:

Def: A torus T over k is a k-group that is iso to a power $(\mathbb{G}_m)^r$ for some $r \geq 0$.

Note: It follows directly from the def that any torus is reductive.

Prop: Let G be a connected solvable linear gp. Then $G = T \ltimes R_u(G)$ scheme theoretically.

Proof: This is a consequence of Lie-Kolchin's Thm see Borel. Linear Alg gps 10.6(4).

Corollary: Tori are exactly the connected solvable reductive groups. In part if G is a linear alg gp, any torus of G is contained in a Borel subgp of G.

Prop: Let G be a connected red grp. Then

① $R(G)$ is a torus

② $Z_G(T)$ is connected and reductive

③ G is a central extension of a connected semi-simple gp by a torus

Proof: ① $R_u(R)$ is characteristic so gives a unip. normal connected subgp of G, hence it must be trivial. Conclude using the above prop on the dec of connected solvable linear gps

② This follows from:

$$R_u(Z_G(T)) = \underbrace{R_u(G) \cap Z_G(T)}$$

WTS it is smooth and connected

In part this intersec: is smooth and connected

such an equality holds to be true for any $H \trianglelefteq G$ BUT $Z_G(T)$ has no good reasons to be normal!

- $Z_G(T)$ is connected follows from Borel. Lin. alg gps 11.2
- $Z_G(T)$ is smooth? See Conrad's notes Lemma 2.2.4.
- $R_u(Z_G(T)) = \underline{R_u(G) \cap Z_G(T)}$?
  - This is a unip. subgp of $Z_G(T)$, stable under $Z_G(T)$-conj hence the equality on points!
  - that it is smooth and connected comes from the fact that it is the scheme of fixed pts for the ac: of T on $R_u(G)$ by inner automorphisms.

The latter is smooth & connected (11)

(Exercise 2.4.7 Conrad's notes)

③ Example 1.1.16 Conrad's notes.

Note that using the following prop one can even show that G connected reductive is actually an extension of a torus by a semi-simple group.

Prop: If G is reductive so are $\mathcal{D}(G)$ & $G/\mathcal{D}(G)$

derived subgroup of G namely the unique smooth (closed) k-subgroup s.t $\mathcal{D}(G)(k)$ is the abstract derived subgp of $G(k)$.

Example: $G = Gl(W)$ is an extension of $\mathbb{G}_m$ by $Sl(V)$:

$$1 \to Sl(V) \to Gl(W) \xrightarrow{\det} \mathbb{G}_m \to 1$$

and a central extension of $PGL(W)$ by $R(G)$

which is a central torus.

Prop: Let G be a connected linear alg group over k. Then (12)

① G is a torus if any $g \in G(k)$ is semi simple

② If G contains no non trivial tori, it is unipotent.

Proof: The proof rests on Borel subgroups for which you should use Linear Alg gps: Borel 11.5 (1) and 11.5.(2).

This has direct (very) practical consequences
A general connected linear alg k-gp G either admits a strictly upper triangular faithful rep or contains a non-trivial k-torus.

Theorem: Let G be a linear algebraic group over k

① For any torus $T' \subset G$, $Z_G(T')$ has finite index in $N_G(T')$ and if G is connected so is $Z_G(T')$

② All max. tori are $G(k)$-conjugate

③ If G is connected and reductive and if T is a max torus in G then $Z_G(T) = T$.

If $g \in G(k)$ is semi-simple then $Z(g)^0$ is reductive

I use here without proof that $Z_G(T^{(1)})$ & $N_G(T^{(1)})$ are rep & of finite pres. (so that we can "recover" them from their k-points. (13)

Proof This is Theorem 1.1.19 in Conrad.

Note: For every connected alg gp, the situa- follows that of Gl_n: $Gl_n(k)$ is covered by $B(k)$ and the subset of semi-simple elements in $G(k)$ is covered by the $T(k)$'s as T varies through max tori. Once again to extend the result focus on solvable G and then use conj. of Borel & Tori to reduce yourself that any $g \in G(k)$ lie in a Borel.

Def. Let G be a linear algebraic group. The centralizer of a max torus is called a Cartan subgroup.

Prop: Cartan subgroups are $G(k)$-conj to each others.
(direct by conj. of tori)

Def. • All Cartan subgps of a connected alg gp have the same dim, called the nilp rank of G

• All max tori of a connected alg gp have the same dim, called the red rank.

Note that if G red conn these two quantities are equal.

A combinatorial approach: roots & coroots (14)

Fix a k-torus T. Note that $\mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$ via $t \mapsto t^n$

Denote $X(T) = \mathrm{Hom}_{k\text{-gps}}(T, \mathbb{G}_m)$ – character lattice "finite free $\mathbb{Z}$-modules of rank $\dim T$" (as $T = \dim(\mathbb{G}_m)^r$)

contravariant in T

$X_*(T) = \mathrm{Hom}_{k\text{-gps}}(\mathbb{G}_m, T)$ – cocharacter lattice

covariant in T

⚠ elem^T of char & cochar are denoted additively e.g. $T \overset{a}{\underset{b}{\rightrightarrows}} \mathbb{G}_m$, $a+b: t \mapsto a(t)b(t)$, $-a: t \mapsto 1/a(t)$

There is a perfect pairing

$$X(T) \times X_*(T) \longrightarrow \mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$$
$$a, \lambda \longmapsto a \circ \lambda$$
$$\begin{matrix} T & & \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{G}_m & & T \end{matrix}$$

(denote $\varphi: X_*(T) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$, $\lambda \mapsto (a \mapsto a \circ \lambda)$) the corresponding iso of $\mathbb{Z}$-mod.

Note that

$$X_*(T) \otimes_{\mathbb{Z}} k^\times \simeq T(k)$$
$$\lambda \otimes c \longmapsto \lambda(c)$$

This really comes from the existence of the perfect pairing as it implies $X_*(T) \otimes_{\mathbb{Z}} k^\times = \mathrm{Hom}(X(T), k^\times)$.

As a k-scheme T can be reconstructed from its character lattice via

$$X_*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m \longrightarrow T$$

$\forall A \in k\text{Alg}$

$$(X_*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m)(A) = X_*(T) \otimes_{\mathbb{Z}} A^\times = \mathrm{Hom}_{\mathbb{Z}}(X(T), A^\times)$$

In other words the functors

$$\left\{\begin{array}{c}\text{Finite free}\\ \mathbb{Z}\text{-mod}\end{array}\right\} \longrightarrow \{\text{tori}\}$$
$$M \longmapsto M^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$$
$$X_*(T) \longleftarrow T$$

are inverse anti-equiv.

Prop: Let T be a torus and V be a finite dim linear rep of T over k. For $a \in X(T)$ denote

$$V_a := \{v \in V \mid t.v = a(t)v \ \forall t \in T(k)\}$$

The a-weight space

Then

$$\bigoplus_{a \in X(T)} V_a \longrightarrow V \text{ is an iso}$$

In other words, there is a bij corresp. between

$$\left\{\begin{array}{c}X(T)\text{-graded}\\ k\text{-vector spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{linear rep}\\ \text{of } T\end{array}\right\}$$

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def: If $V_a \neq 0$ then a is a weight of the representation.

Lemma: $\forall\, a \in X(T)$ one has $(V^*)_a = (V_{-a})^*$ where V^* is the dual rep space. In part the set of weights of a self dual rep of T is stable under nega-

E.g.: $T = \mathbb{G}_m$, $X(T) = \mathbb{Z}$

and linear rep of $\mathbb{G}_m$ $\overset{1:1}{\longleftrightarrow}$ $\mathbb{Z}$-graded vector spaces

$$V = \bigoplus_{n \geq 0} V(n)$$

as $t \in \mathbb{G}_m \curvearrowright V(n)$ via t^n-scaling