Grothendieck topologies, torsors and descent

A first example:

Consider the two real Lie alg.

- $sl_2(\mathbb{R}) = \{M \in M_2(\mathbb{R}) \mid tr(M) = 0\}$ that can be viewed as a subspace of $sl_2(\mathbb{C})$ via
  $sl_2(\mathbb{R}) = \{M \in sl_2(\mathbb{C}) \mid \bar{M} = M\}$
- $su_2(\mathbb{R}) = \{M \in sl_2(\mathbb{C}) \mid \bar{M} = -M^t\}$.

These are non isomorphic real Lie alg, but they are "locally" iso in the sense that

$sl_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong su_2(\mathbb{R}) \otimes \mathbb{C}$

In the same way $SL_{2,\mathbb{R}}$ and $SU_{2,\mathbb{R}}$ are

$SL_{2,\mathbb{C}}$ cut by the equation $\bar{M} = (M^t)^{-1}$

not isomorphic but become isomorphic after base change to $\mathbb{C}$

? What do I mean when I write "locally"?

From now on k is any (commutative, unital) ring.

Remember that an affine scheme X is a locally ringed space $(X, \mathcal{O}_X)$

the structure sheaf of the topological space X

Namely, to any open subset $\mathcal{U} \subset X$ and any inclusion map $\mathcal{U} \subset \mathcal{V}$, $\mathcal{O}_X$ associates an object $\mathcal{O}_X(\mathcal{U})$ (here a local ring) and a restric° map $res_{\mathcal{V},\mathcal{U}}$ (here a morphism of local rings) that satisfies the axioms you're expecting (namely $res_{\mathcal{U},\mathcal{U}} = id$, $res_{\mathcal{V},\mathcal{U}} \circ res_{W,\mathcal{V}} = res_{\mathcal{U},W}$ $\forall \mathcal{U} \subseteq \mathcal{V} \subseteq W$) + others, called the glueing axioms

These are made to bound together local and global datas namely $\mathcal{O}_X(\mathcal{U})$ and $\mathcal{O}_X(X)$ for $\mathcal{U}$ a "small" open subset.

? This is what we need to formalize within the categorical approach we've followed since the 1st lecture.

Roughly speaking we need to:

- Introduce the no° of topology on a category
- This being done we want to be able to cover an object $\mathcal{U}$ by a family $(\mathcal{U}_i)_{i\in I}$ of "open neighborhoods
- A last hope there would be that to get a property (globally) on $\mathcal{U}$ it will be enough to check it locally (on the $\mathcal{U}_i$'s)

Example: X = topological space

We associate to X the category $\hat{X}$ whose objects are all open subsets of X and whose morphisms are inclusions of open sets. If $\mathcal{U} \subset X$ is open then

1. $\mathcal{U}$ is a covering of itself
2. If $(\mathcal{U}_i)_{i\in I}$ is a covering of $\mathcal{U}$ and if for each $i \in I$, $(\mathcal{U}_{ij})_{j\in J}$ is a covering of $\mathcal{U}_i$ then $(\mathcal{U}_{ij})_{i\in I, j\in J}$ is a covering of $\mathcal{U}$
3. If $\mathcal{V} \subset \mathcal{U}$ is an open subset and $(\mathcal{U}_i)_{i\in I}$ is a covering of $\mathcal{U}$, then $(\mathcal{U}_i \cap \mathcal{V})_{i\in I}$ is a covering of $\mathcal{V}$.

In categorical terms for $\hat{X}$, a morphism $\mathcal{V} \to \mathcal{U}$ is the inclusion of $\mathcal{V}$ in $\mathcal{U}$. A covering is then a family of morphisms $\mathcal{U}_i \to \mathcal{U}$ in $\hat{X}$. Now, the intersec° of two open subsets V and W of $\mathcal{U}$ is nothing but the pullback (exercise!)

$$\begin{tikzcd} V \times_{\mathcal{U}} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d] \\ W \arrow[r] & \mathcal{U} \end{tikzcd}$$

The categorical analogues of 1...3 then writes

C1 If $\mathcal{V} \to \mathcal{U}$ is an iso then $\mathcal{V} \to \mathcal{U}$ is a covering of $\mathcal{U}$.

C2 Same as 2

(C3) For each morphism $\mathcal{V} \to \mathcal{U}$ and each covering $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{U}$, the family $(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{V})_{i \in I}$ is a covering of $\mathcal{V}$.

This starting point is to be generalized to other categories:

If $\mathcal{C}$ is a category and $\mathcal{U} \in \mathrm{Ob}(\mathcal{C})$, a covering of $\mathcal{U}$ is any family of morphisms $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$

Grothendieck topologies are def from coverings that behave well.

Def. • Let $\mathcal{C}$ be a category. A Grothendieck topology on $\mathcal{C}$ consists, for each $\mathcal{U} \in \mathcal{C}$ of a collection $\mathrm{cov}(\mathcal{U})$ whose elements are coverings of $\mathcal{U}$ s.t C1 - C3 hold (where C3 is to be understood that pullbacks exist in $\mathcal{C}$).

• A site is a category $\mathcal{C}$ endowed with a topology T

We write $\mathcal{C}_T$ such a site.

Example (following the preceeding example).

For each topological space X, $\hat{X}_{open}$ is a site where Open is the topology whose coverings are all open coverings.

Grothendieck topologies on the cat of affine schemes

We stick to affine schemes as we've limited ourselves to this setting in the framework of this course but what follows easily extends to schemes.

(1) The Zariski topology.

Each affine k-scheme $\mathcal{U} = \mathrm{Spec}\, A$ is a topological space under the Zariski topology

⇝ the Zariski topology on the category of affine k-schemes is given by letting each $\mathrm{cov}(\mathcal{U})$ consist of all open coverings of $\mathcal{U}$ by Zariski-open immersions.

From a functor of points point of view, $\mathcal{U} \subseteq \mathrm{Hom}_A$ is open if $\exists$ an ideal $J \subseteq A$ st $\forall R \in k\text{-Alg}$

$$\mathcal{U}(R) = \{ \Phi \in \mathrm{Hom}_A(R) \mid R\Phi(J) = R \}$$

(which amounts to saying that $\mathcal{U}$ corresponds to the complement of the zero locus of J in $\mathrm{Spec}(A)$).

An open immersion $i : \mathcal{Y} \to \mathrm{Hom}_A$ is then a monomorphism s.t the image functor is an open subfunctor of Hom_A

Hence $(\mathcal{U}_i \to \mathcal{U})_i \in \underbrace{\mathrm{Cov}(\mathcal{U})}_{\text{for the Zariski top}}$ when $\mathcal{U}_i \to \mathcal{U}$ is an open immersion

and the $\mathcal{U}_i \to \mathcal{U}$'s are jointly surj

namely $\mathcal{U}_i = \mathrm{Hom}_{A_i}$

$\mathcal{U} = \mathrm{Hom}_A$

and $\forall \mathfrak{p} \subset A$ prime $\exists i \in I$ and $\mathfrak{q} \subset A_i$ s.t $\mathfrak{p} = \varphi_i^{-1}(\mathfrak{q})$

Small vs big site:

$(k\text{Aff-sch})_{Zar}$ def above is referred as the big Zariski site of Affine k-schemes as, as a category, we consider the whole category k-Aff.

The small Zariski site is obtained by restricting ourselves to consider only morphisms compatible with the top.

The terminology big vs small site generalize to any other topology.

If k is a field and $U = \mathrm{Hom}_k$ there is no non trivial Zariski covering of U (as there are no non zero proper ideals) ... ⇝ This calls for stronger / finer topologies

Def. If J and J' are two Grothendieck topologies on a category $\mathcal{C}$, J is finer / stronger than J' if every J'-cov is a J-cov. In this case J' is coarser than J.

② The étale topology

def A morphism $X = \mathrm{Hom}_A \longrightarrow S = \mathrm{Hom}_B$ is étale if X is a finitely presented B-scheme and if for each $R \in B\text{-Alg}$ & any $J \trianglelefteq R$ ideal s.t. $J^2 = 0$ the canonical map $X(R) \longrightarrow X(R/J)$

is bij (note that for smoothness it was asked to be surj)

Ex - Prop: Let F be a field. $A \in F\text{-Alg}$ is étale iff A is a product of finite separable field extensions of F.

As a consequence a field ext. is étale iff it is finite and separable.

Fiberwise criterion for étale morphisms:

Let $f: X \longrightarrow S$ morphism, X into a flat B-scheme of finite presentation.

f is étale iff X_F is étale $\forall\, F \in B\text{-Alg}$ that is a field.

Def: The étale topology on k-Aff is the data of $(U_i \longrightarrow U)_i \in \mathrm{Cov}(U)$ with $U_i \longrightarrow U$ étale and $U_i \longrightarrow U$'s are jointly surjective.

The following topology, which will be our favourite allows one to release the separability condi^2 which appears to be too strict in some situations

③ The fppf-topology (faithfully flat of finite pres)

Remember that $\mathrm{Hom}_A \longrightarrow \mathrm{Hom}_B$ is flat when

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$\forall (M,N) \in B$-Mod and $f: M \to N$, a B-module homomorphism, if $f: M \to N$ is injective then so is the map $Id_A \otimes f: A \otimes_B M \to A \otimes_B N$.

It is faithfully flat if the above implica^n is an equiv $\forall (M,N) \in B$-Mod.

Any flat group scheme is faithfully flat.

Def: The fppf topology on k-Aff is given by coverings $(U_i \to U)$ where each $U_i \to U$ is flat of finite presentation and the morphisms $U_i \to U$ are jointly surj.

Example: Every field extension is faithfully flat, so if E and F are fields then $Hom_E \to Hom_F$ is an fppf-cov iff E is a finite field extension of F.

One can weaken even more the above condi^n by replacing the finite presenta^n requirement with a weaker one:

④ fpqc-topology (faithfully flat & quasi-compact)

Def A topological space is quasi-compact if it is compact with no Hausdorff assump^n. Viewing schemes geometrically as topological spaces under the "classical" Zariski topology, a morphism is quasi compact if each quasi-compact open subset of the target is the image of a quasi-compact open subset.

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Def The fpqc topology on k-Aff is given by coverings $\{U_i \to U\}_{i \in I}$ s.t $U_i \to U$ is flat $\forall i \in I$, the induced map $\bigsqcup_{i \in I} U_i \to U$ is quasi-compact and, for each $R \in k$-Alg $U(R)$ is the union of the image of all $U_i(R)$.

Exercise Show that on k-Aff, fpqc is stronger than fppf, which is stronger than étale, which is stronger than Zariski and that these inclusions are strict, for instance $\exists$ an fpqc covering that is not an fppf covering etc.

Sheaves.

Remember the geometric descrip^n of a scheme as a pair $(X, \mathcal{O}_X)$

a topological space

its structure sheaf, to be thought as a contravariant functor from the category whose objects are the open subsets of X & whose morphisms are the inclusion maps $V \to U$ of open sets of X, to the cat. of sets that satisfies the so-called gluing axioms that are made to bind local and global datas

? How does this apply to Grothendieck's topologies?

Let $\mathcal{C}$ be a category

Def: A presheaf on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to the category of sets

Note: This doesn't involve the no° of site. By the Yoneda correspondence, a presheaf on k-Aff is precisely a k-functor (i.e a covariant functor from k-Alg to Set). By changing the target category we get presheaves of groups, of rings...

Example: Let X be a topological space, a presheaf on X (in the classical sense) agrees with the def of a presheaf on the site $\hat{X}_{open}$ whose coverings are open coverings

Def: Let $\mathcal{C}_{\mathcal{J}}$ be a site. A presheaf $\mathcal{F}$ on $\mathcal{C}$ is a sheaf for the topology $\mathcal{J}$, also called a $\mathcal{J}$-sheaf, if $\forall U \in \mathcal{C}$ and $\forall (U_i \to U)_{i\in I} \in \mathcal{J}$-cov, the diag

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact, namely the first arrow is inj and its image is equal to the $\underbrace{\text{kernel of the double arrow}}_{\text{i.e elements of the middle term whose two images on the right agree}}$

Examples: ① Let X be a topological space. In the classical setting a presheaf is a sheaf if it satisfies the gluing axioms:

$\forall U \subseteq X$ and any open cover $(U_i)_{i\in I}$ of U the diagram below, induced by applying $\mathcal{F}$ to the inclusion maps, is exact.

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

Hence a sheaf on X (in the classical sense) is the same as a sheaf on $\hat{X}_{open}$.

② A presheaf on $(k\text{-Aff})_{fpqf}$ is a covariant functor $\mathcal{F}: k\text{-Alg} \longrightarrow \text{Set}$

If $R \in k$-Alg and $A \longrightarrow B$ is a faithfully flat morphism of R-modules, then the axioms translate to

$$\mathcal{F}(A) \longrightarrow \mathcal{F}(B) \rightrightarrows \mathcal{F}(B \otimes_A B)$$

where the double arrows are the images under $\mathcal{F}$ of $- \otimes 1_B$ and $1_B \otimes -$ respectively.

Thus a section in $\mathcal{F}(B)$ is def over A precisely if it maps to the same thing under those maps.

Compare this to the fact that one can slip an element from one side of $B \otimes_A B$ to the other precisely if it belongs to A.

Remark: (Sheaves on k-Aff) From the above comparison of topologies on k-Aff deduce that every fpqc-sheaf is an fppf-sheaf, every fppf-sheaf is an étale sheaf and every étale sheaf is a Zariski sheaf.

⚠ Even if a topology is strictly finer than another the topology of sheaves wrt the two topologies may still coincide.

? Sheaves on k-Aff and representable k-functors are both nice classes of covariant functors $k\text{-Aff} \to \text{Set}$. Can we compare them?

Propos: (to be shown latter)
Every representable k-functor is an fpqc sheaf.

Note: ① The proof requires descent theory
② This implies that representable functors are also sheaves wrt fppf, étale & Zariski top.
This also gives us a practical necessary condi° to check whether a functor is representable or not.

Sheafification & discussion on quotients.

As already mentioned in many occasions during the lectures many construc° made in the classical setting (for $k = \bar{k}$ alg closed) may be challenging to be

representable in the relative setting.
The first thing we need to check then is that these arise as sheaves.

? What can we do if we start with a construc° that is not a sheaf?

Thm-Def: Let $\mathcal{C} = k$-Aff and $J \in \{\text{fppf, étale, Zariski}\}$
To $\underbrace{\text{each presheaf}}_{\text{ou } k\text{-functor}}$ $\mathcal{F}$ on $\mathcal{C}$ corresponds a unique J-sheaf $\bar{\mathcal{F}}$ together with a natural transforma° $\alpha : \mathcal{F} \to \bar{\mathcal{F}}$ s.t.:

$\forall$ J-sheaf $\mathcal{G}$
$\forall$ natural transf. $\eta : \mathcal{F} \to \mathcal{G}$, $\exists ! \ \bar{\eta} : \bar{\mathcal{F}} \to \mathcal{G}$
making the following diag commute

$$\begin{tikzcd} \bar{\mathcal{F}} \arrow[r, dashed, "\bar{\eta}"] & \mathcal{G} \\ \mathcal{F} \arrow[u, "\alpha"] \arrow[ur, "\eta"'] & \end{tikzcd}$$

such a pair $(\bar{\mathcal{F}}, \alpha)$ is called the J sheaf associated to $\mathcal{F}$ or the **sheafification** of $\mathcal{F}$ wrt J

"Idea of proof" Let $\mathcal{F}$ be a presheaf we def a presheaf $\mathcal{F}'$ as follows: $\forall \mathcal{U} \in \mathcal{C}$, $\mathcal{F}'(\mathcal{U}) = \lim\limits_{(\mathcal{U}_i \to \mathcal{U})_{i\in I} \in \mathrm{Cov}(\mathcal{U})} \left(\prod \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j) \right)$

and repeat the process once.

Reminder on direct limits.

Let $(I, \leq)$ be a directed set (namely a preordered set in which every finite subset has an upper bound.

A directed system over I is a family $\{A_i : i \in I\}$ of objects indexed by I together with morphisms $f_{ij} : A_i \longrightarrow A_j$ $\forall\, i \leq j$ such that f_{ii} is the identity on A_i and $\forall\, i \leq j \leq k$ one has $f_{ik} = f_{jk} \circ f_{ij}$.

Let $\mathcal{C}$ be a category and $\langle A_i, f_{ij} \rangle_{i \in I}$ a direct system of objects and morphisms in $\mathcal{C}$.

The direct limit of the directed system $\langle A_i, f_{ij} \rangle$ is a pair $(\varinjlim A_i, \varphi_i) \in \mathrm{Ob}\,\mathcal{C} \times (\mathrm{Mor}\,\mathcal{C})^I$ s.t $\varphi_i : A_i \longrightarrow \varinjlim A_i$ $\forall\, i \in I$ and $\varphi_i = \varphi_j \circ f_{ij}$ whenever $i \leq j$, that satisfies the following universal property

$\forall\, (Y, \psi_i) \in \mathrm{Ob}\,\mathcal{C} \times (\mathrm{Mor}(\mathcal{C}))^I$ such that $\psi_i : A_i \longrightarrow Y$ and $\psi_i = \psi_j \circ f_{ij}$ $\forall\, j \leq i$, $\exists!\, u : \varinjlim A_i \longrightarrow Y$ such that the following diag commute

$$\begin{tikzcd}
A_i \arrow[rr, "f_{ij}"] \arrow[dr, "\varphi_i"] \arrow[ddr, bend right, "\psi_i"'] & & A_j \arrow[dl, "\varphi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& \varinjlim A_i \arrow[d, dashed, "u"] & \\
& Y &
\end{tikzcd}$$

In the case we are interested in:

if $\mathcal{U} \in \mathcal{C}$, let $\Sigma_{\mathcal{U}}$ the set of open coverings $\mathcal{V} = \{\mathcal{V}_i\}_{i \in I}$ where $\mathcal{V}_i \neq \mathcal{V}_j$ $\forall\, i \neq j$ together with the order given by refinements

$$\mathcal{F}'(\mathcal{U}) = \varinjlim_{\mathcal{V} \in \Sigma_{\mathcal{U}}} D(\mathcal{V})$$

$$D(\mathcal{V}) = \left\{ (s_i) \in \prod_{i \in I} \mathcal{F}(\mathcal{V}_i) \;\middle|\; s_i\big|_{\mathcal{V}_{ij}} = s_j\big|_{\mathcal{V}_{ij}} \in \mathcal{F}(\mathcal{V}_i \times_{\mathcal{U}} \mathcal{V}_j) \right\}$$

This is a directed system

image of $\mathcal{V}_i \times_{\mathcal{U}} \mathcal{V}_j$ in $\mathcal{V}_i$, resp $\mathcal{V}_j$

We are considering direct limits here that may cause set-theoretical problems, this can be fixed by choosing a universe BUT the limit may not exist in that universe: this happens already for the fpqc sheaves, namely there exist fpqc presheaves that are not fpqc sheaves.

This problem doesn't occur in the case of Zariski, étale and fppf topologies because of the finiteness conditions involved in the def of these topologies. In other words associated sheaves always exist and are independent of any choice of universe.

Remark: Taking twice the limit?

The first limit allows one to get an injective morphism

$$\mathcal{F}'(\mathcal{U}) \longrightarrow \prod_i \mathcal{F}'(U_i)$$

but one needs to take the limit again to ensure that the image lands in the kernel of the corresponding double arrow