Algebraic groups - Lecture 6

(1)

Remember: * Let $G = \mathrm{Hom}_A$ be an affine k-grp scheme

We associate to G its tgt space at identity

$Lie(G) = T^e_G := V_{\varepsilon^*\Omega_A} \longrightarrow V_{\Omega_A} \cong G(\_[\varepsilon])$

$= \ker(G(\_[\varepsilon]) \to G)$

$\downarrow \qquad \downarrow$

$\mathrm{Hom}_k \longrightarrow \mathrm{Hom}_A$

- inherits an additive grp structure from V_{Ω_A}
- $\forall R \in k\text{Alg}$ a way to view elements of $Lie(G)(R)$ as the "ε-coeff" of elements in $G(R[\varepsilon])$
- $\forall R \in k\text{-Alg}$, $Lie(G)(R) = Der_A(A, R^e)$ endowed with an A-module structure induced by $e: A \to k$.

module of Kähler differentials (Ω_A)

- where $\varepsilon = [x]$ in $R[x]/(x^2)$

and

- $G(\_[\varepsilon]): k\text{Alg} \longrightarrow \text{Grp}$, $R \longmapsto G(R[\varepsilon])$

- $\forall R \in k\text{Alg}$ $V_{\Omega_A}(R) = \mathrm{Hom}_{R\text{-Mod}}(R \otimes \Omega_A, R)$ representable

* We define the Lie functor

$Lie: \text{Aff-Grp-sch} \longrightarrow \text{Lie-Alg-sch}$

$G \longmapsto Lie(G)$

$F: G \to H \longmapsto Lie(F)$ def $\forall R \in k\text{-Alg}$

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by the commutativity of the following diag:

$$\begin{array}{ccccccccc} 1 & \to & Lie(G)(R) & \to & G(R[\varepsilon]) & \xrightarrow{G(p_R)} & G(R) & \to & 1 \\ & & \downarrow{\scriptstyle Lie(F)} & & \downarrow & & \downarrow & & \\ 1 & \to & Lie(H)(R) & \to & H(R[\varepsilon]) & \to & H(R) & \to & 1 \end{array}$$

where $p_R: R[\varepsilon] \to R$

To get a Lie alg structure on $Lie(G)(R)$ one needs:

- a scalar action: if $R \in k\text{Alg}$ and $a \in R$

def $u_a: R[\varepsilon] \longrightarrow R[\varepsilon]$

$r + \varepsilon s \longmapsto r + a\varepsilon s$

This defines a homomorphism

$u_R: G_a \longrightarrow End(Lie(G))$ s.t. $\forall R \in k\text{Alg}$

$u_R: G_a(R) \longrightarrow End(Lie(G)(R))$

$a \longmapsto G(u_a)$

Denote $G(u_a)(x) =: ax$

- a Lie bracket.

$Ad: G \longrightarrow GL(Lie(G))$ def $\forall R \in k\text{Alg}$ via

$Ad(R): G(R) \longrightarrow GL(Lie(G))(R)$

$g \longmapsto Ad_g: Lie(G)(R) \longrightarrow Lie(G)(R)$

$g \longmapsto Ad_g$

Applying the Lie functor one gets

$$\mathrm{ad} : \mathrm{Lie}(G) \longrightarrow L(\mathrm{Lie}(G))(R)$$

where, $\forall R \in k\mathrm{Alg}$, $\mathrm{ad}_R : x \longmapsto \mathrm{ad}_x$

and we define, $\forall x, y \in \mathrm{Lie}(G)(R)$

$$[x,y] = \mathrm{ad}_x(y)$$

$\cong \mathrm{Lie}(GL(\mathrm{Lie}(G)))(R)$

Remember that if $M \in k\mathrm{Mod}$

$L(M) : k\mathrm{Alg} \longrightarrow \mathrm{Mon}$

$R \longmapsto \mathrm{Hom}_{R\text{-}\mathrm{Mod}}(M \otimes R, M \otimes R)$

Exponential notation

To deal with the notational confusion of having an additive grp law in $\mathrm{Lie}(G)(R)$ but a mult. one in $G(R[\varepsilon])$, one needs to transform addition into multiplication as we move between the two ones. This is achieved by the exp notation.

Let $R \in k\mathrm{Alg}$. For any $S \in k\mathrm{Alg}$ with $\alpha \in S$ s.t $\alpha^2 = 0$ there is a unique R-alg. hom.

$R[\varepsilon] \longrightarrow S$. By functoriality of G we get

$\varepsilon \longmapsto \alpha$

the composite map

$$\mathrm{Lie}(G)(R) \longrightarrow G(R[\varepsilon]) \longrightarrow G(S)$$

$$x \longmapsto e^{\alpha x}$$

In particular, for $S = R[\varepsilon]$, $\alpha = \varepsilon$ one gets

$$e^{\varepsilon x} = \mathrm{Id} + \varepsilon x .$$

⚠ Note that while this is indeed the start of the Taylor expansion of the usual exponential function and while over fields of char 0 (or p under carefully chosen assump°) this has an interpreta° in terms of exponentials, this is not the case in general.

IT IS JUST A PIECE OF SUGGESTIVE NOTA°.

Remark: $\forall R \in k\mathrm{Alg}$, $S \in R\mathrm{Alg}$ and $\alpha \in S$ s.t $\alpha^2 = 0$ one has:

① $e^{\alpha x} e^{\alpha y} = e^{\alpha(x+y)}$ for any $x, y \in \mathrm{Lie}(G)(R)$. Indeed $x + y \in \mathrm{Lie}(G)(R)$ writes $\mathrm{Id} + \varepsilon(x+y)$ in $G(R[\varepsilon])$ using the identification $R[\varepsilon] = R + \varepsilon R$ as R-module

$$\mathrm{Id} + \varepsilon(x+y) = (\mathrm{Id} + \varepsilon x)(\mathrm{Id} + \varepsilon y) \quad \text{because } \varepsilon^2 = 0$$
$$= e^{\varepsilon x} e^{\varepsilon y}.$$

② For any group homomorphism $F : G \longrightarrow H$ one has $F(e^{\alpha x}) = e^{\alpha \mathrm{Lie}(F)(x)}$ $\forall x \in \mathrm{Lie}(G)(R)$.

③ $e^{\alpha(ax)} = e^{(a\alpha)x}$ $\forall a \in R$, $x \in \mathrm{Lie}(G)(R)$.

④ $\forall R \in k\mathrm{Alg}$, as $\mathrm{ad}_x := \mathrm{Lie}(\mathrm{Ad}_{\varepsilon x})$ one gets

$$\mathrm{Ad}_{\varepsilon x} = \mathrm{Id} + \varepsilon\, \mathrm{ad}_x,$$ namely $\forall R \in k\mathrm{Alg}$ and $x, y \in \mathrm{Lie}(G)(R)$,

$$\mathrm{Ad}_{\varepsilon x}(y) = y + \varepsilon[x,y].$$

Lemma: $\forall R \in k\text{Alg}$, $S \in R\text{Alg}$ and $\alpha, \beta \in S$ with (5)
$\alpha^2 = \beta^2 = 0$ the identity

$$e^{\alpha x} e^{\beta y} e^{-\alpha x} e^{-\beta y} = e^{\alpha\beta [x,y]}$$ holds.

Proof: Exercise!

Using what preceeds one shows that

Proposition: $\forall R \in k\text{Alg}$, the set $\mathcal{L}ie(G)(R)$ endowed with the additive law inherited from the mult law on $G(R[\varepsilon])$, scaling and bracket as def above is an R-Lie alg.

Summary: For any affine k-alg grp G we have defined a k-functor

$$\mathcal{L}ie(G) : k\text{Alg} \longrightarrow k\text{-Lie alg}$$
$$G \longmapsto \mathcal{L}ie(G)(R)$$

From the Lie functor $\mathcal{L}ie(G)$ one gets the Lie alg of G as follows

Def: The Lie alg of G is the k-Lie alg

$$\text{Lie}(G) := \mathcal{L}ie(G)(k)$$

⚠ As noted during the last lecture it is not true in general that, given $R \in k\text{Alg}$

$$\mathcal{L}ie(G)(R) = \mathcal{L}ie(G)(k) \otimes R$$

Otherwise stated (6)

$$\text{Lie}(G_R) \not\cong \text{Lie}(G) \otimes_k R \quad \text{in general.}$$

The above inequality is an equality iff $\Omega_{G/k}$ is loc free of finite type

This is in part satisfied when G is smooth over k or when k is a field and G is loc. of finite presenta° over k

In part: When G is of finite presentation

$$\text{Lie}(G) \cong \text{"Lie}^{\text{diff}}(G)\text{" as } R\text{-Lie algebras.}$$

Remark: Saying that $\forall R \in k\text{Alg}$ one has $\text{Lie}(G_R) \cong \text{Lie}(G) \otimes_k R$ amounts to say that $W_{\text{Lie}(G)} \to \mathcal{L}ie(G)$ is an iso of scheme

rep iff $\text{Lie}(G)$ is finitely gen. & proj

Remember: $\forall M \in k\text{Mod}$, $\left(W_M : k\text{-Alg} \longrightarrow \text{Grp},\ R \longmapsto R \otimes_k M \right)$

Namely, $\text{Lie}(G)$ is sufficient to fully det. $\mathcal{L}ie(G)(R)$ $\forall R \in k\text{Alg}$.

Examples:

① The Lie alg of the additive grp.

$\forall R \in k\text{Alg}$, $\text{Lie}(G_a)(R) = W_k(R)$

(exercise! Use the exact sequence

$$1 \to \text{Lie}(G_a)(R) \to G_a(R[\varepsilon]) \to G_a(R) \to 1$$

and the def of G_a as functor of points)

So $\text{Lie}(G_a) = \text{Lie}(G_a)(k) = W_k(k) = k$.

② The Lie alg of G_m.

$\forall R \in k\text{Alg}$, show that

$$(R[\varepsilon])^* = \{a + \varepsilon b \mid a \in R^*, b \in R\}$$

hence $\text{Lie}(G_m)(R) = W_k(R)$

and $\text{Lie}(G_m) = k$.

③ The Lie algebra of GL_n:

$\forall R \in k\text{Alg}$ show that any $M \in GL_n(R)$ dec as $M = A + \varepsilon B$ with $A \in GL_n(R)$ and $B \in M_n(R)$

hence $\text{Lie}(GL_n) = M_n(k)$.

Lie algebras and smoothness.

We have seen before that smoothness is essentially def by reaching to the field case and demanding that the tangent space at a point have the same dimension as the scheme at that point. Using Lie algebras this can be phrased more succinctly as follows.

Thm: Assume that k is a field and let $G = \text{Hom}_A$ be an affine k-grp scheme of finite type. Then G is smooth if and only if

$$\dim_k \text{Lie}(G) = \dim(G)$$

(dim as a k-vs; Krull dim of Spec A; one always has $\geq$)

Remark: As mentioned last time, the Lie functor is left exact but not right exact in general. When k is a field right exactness is ensured by smoothness. [D&G, II §5 n° 5.3. Proposition]

If G is smooth over k, let $f: G \to H$ be a morph of k-grps of finite pres.

$\text{Lie}(f): \text{Lie}(G) \to \text{Lie}(H)$ is surj $\iff$ H is smooth and f is smooth

$\iff$ $\ker(f)$ is smooth, $f(G)_{red}$ is open in H.

(reduced subscheme of $f(G)$, $X = \text{Spec}\, B$ is reduced if B is (has no nilp) non-zero elem)

Reductive groups over algebraically closed fields

In what follows k is an alg closed field

In this case, let X be a finitely generated scheme over k, then $X(k)$ is Zariski dense in X.

This comes from: ① The density of closed points (that holds to be true for loc. finitely generated schemes over any field, see for instance Görtz-Wedhorn Prop 3.35)

② For alg closed fields k-rational points of a scheme loc. finitely gen over k, are the closed points (see for instance Görtz-Wedhorn §5.1).

⇒ Most of the time, in the classical setting properties will be def on points!

Def: Linear algebraic group

A linear alg. group over $k = \bar{k}$ is a smooth affine k-group scheme

The terminology comes from the fact that such groups are precisely the smooth k-groups G that are finitely generated and for which there exists

a k-homomorphism

$$j : G \hookrightarrow GL_n$$

that is a closed immersion.
(see for instance Borel. Linear alg groups 1.10. Proposition).

? Subgroups ?

Proposition. Let G be a linear alg grp and $H \subseteq G$ be a k-subgroup. Then H is a closed subgroup of G (so we don't need to bother with representability issues).

• In part. if a linear representation is faithfull (namely if it has trivial kernel) then it is a closed immersion.

Proof. This is a direct consequence of Prop 1.1.1 in Conrad's notes:

If $f: G \to H$ is a morphism of lin. alg k-gps and if $\ker(f)(k)$ is finite then f is finite flat onto its smooth closed image and if $\ker f$ is trivial then $\ker f$ is a closed immersion.

Solvable and reductive groups

Let G be a linear alg group and fix a faithful linear representation (namely it has trivial kernel)

$$j : G \hookrightarrow GL(V)$$

Def: For any $g \in G(k)$ we say that g is

- semi simple if the linear endomorphism $j(g)$ is diagonalizable (that is, semi-simple in the sense of linear alg)
- unipotent if $j(g)$ is unipotent as a linear endomorphism of V.

Remark: These properties are independent of the choice of the rep j and are preserved under any k-hom. $f: G \to H$ of lin. alg k-groups.

Jordan dec: $\forall g \in G(k)$ $\exists! g_{ss}, g_u \in G(k)$ s.t

- g_{ss} is semisimple
- g_{u} is unipotent
- $g = g_{ss} g_u = g_u g_{ss}$

This decomposition is stable under homomorphisms of linear alg k-grps: let $f: G \to H$ be such a morphism.

Then $f(g)_{ss} = f(g_{ss})$ and $f(g)_u = f(g_u)$.

Def: (Unipotent group)

A linear alg group G is unipotent if $g = g_u$ $\forall g \in G(k)$.

example: G_a is ① A linear alg group as it is smooth (use the lie alg criterion as done to get convinced).

② unipotent:

Consider $j: G \to GL_2$ s.t $j_k : G(k) \hookrightarrow GL_2(k)$

$$g \mapsto \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}$$

it is a faithful representa° and $j_k(g)$ is unip. $\forall g \in G(k)$.

Actually a linear algebraic k-group G is unip. precisely when it occurs as a closed subgroup of the subgroup of strictly upper triang. matrices $U_n \subseteq GL_n$ for some $n \in \mathbb{N}$.

Note that U_n admits a composi° series (def on the abstract group $U_n(k)$) (normal series where each factors is simple) whose successive quotients are G_a.

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Hence:

Thm: Let U be a k-linear alg gp. TFAE:

1) U is unipotent.

2) U has a composition series whose successive quotients are $\mathbb{G}_a$ if $char(k) = 0$ or $\mathbb{G}_a$, α_p or $\mathbb{Z}/p\mathbb{Z}$ if $char(k) > 0$.

3) U is isomorphic to a linear alg subgrp of U_n.

Remarks:

(a) This is Grothendieck's def of unipotent groups which has the advantage of avoiding the choice of a GL_n-embedding (see SGA 3, XXVII)

(b) In part. if $char(k) = 0$ any unip. grp is connected.

Prop: 1) Every closed subgroup of a unipotent linear alg group is unip and every quotient of a unip. linear alg group is unip.

2) If G is a linear alg k-gp s.t. $\exists\, U \trianglelefteq G$ normal unip. and G/U is unip. then so is G.

Proof: Comes from the functoriality of the Jordan dec

Unip. lin alg k-groups are a particular ex of

? Note on quotients.

Let $H \leq G$ be two linear alg k-gp

- Classically a smooth quasi-proj quotient G/H is built so that $q: G \to G/H$ identifies $(G/H)(k)$ with $G(k)/H(k)$ and on the tangent spaces $T_{e}(q)$ is the natural surjection $Lie(G) \twoheadrightarrow Lie(G/H)$

- Scheme-theoretically the quotient morph $G \to Q$ by the right H-action on G is uniquely characterized (if it exists!) by the condi: that it represents the fppf-sheafification of the functor $S \to G(S)/H(S)$ on the cat of k-schemes.

We will see in a few weeks that these def coincide here

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Def: Solvable groups:

① An abstract group Γ is solvable if its derived series $G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \dots$ terminates, where $G^{(0)} = G$ and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$

• A linear alg. k-group G is solvable if $G(k)$ is solvable as an abstract group.

Thm (Lie-Kolchin): If G is solvable and (connected) every linear representa° $G \longrightarrow GL_n$ can be conjugated to have its image inside the upper triangular subgroup B_n.

↑ for $G = Hom_A$

A is non zero and has no non-trivial idempotent

$\Longrightarrow$ In particular every unip. linear alg group is solvable.

Prop: Let G be a k-lin alg group. There exist:

1) A max unip k-normal linear alg subgrp of G, denoted $R_u(G)$ and called the unip radical of G

2) A max connected solvable normal linear alg subgrp $R(G)$ called the radical of G.

Proof ① Key fact

If $U, U' \trianglelefteq G$ are connected normal unip linear alg subgps of G then $U.U' \trianglelefteq G$ is closed and unip (it is a quotient of $U \rtimes U'$)

Def in a scheme theoretic way to be the semi-direct pdct on R-pts $\longrightarrow$ $\forall R \in kAlg$

One concludes by dimension considera°.

② similar

Prop: If $H \trianglelefteq G$ is a normal linear alg subgp then $R_u(H) = (H \cap R_u(G))^{\circ}_{red}$

$R(H) = (H \cap R(G))^{\circ}_{red}$

⚠ The intersec° of 2 smooth schemes need not be smooth!

where for a group scheme $\mathcal{G}$, $\mathcal{G}^{\circ}$ is the connected component of the identity (it is representable!) and $\mathcal{G}_{red}$ is the reduced (= smooth here!) subscheme of $\mathcal{G}$.

Note that the forma° of the underlying reduced scheme is local for the Zariski topology so both forma° commute here, namely

$$(\mathcal{G}^{\circ})_{red} = (\mathcal{G}_{red})^{\circ}$$

Proof: $R_u(H)$ is normal in G.

Prop: If $\pi: G \to H$ is a surj hom of linear alg k-grps then $R_u(H) = \pi(R_u(G))$ and likewise for radicals.

Proof: Linear alg grps. Borel 14.11.

Ex: 1) $R(GL_n)$ = scalar matrices.

2) $R_u(B_n) = U_n$

Def: • A red k-group is a linear alg k-grp G s.t $R_u(G) = \{1\}$

• A semi-simple k-gp is a lin. alg k grp s.t $R(G) = \{1\}$.

Examples: ① $GL(V)$ is a connected red gp.

Gm

② Semi-simple gp? SL_n.