Grothendieck topologies

There is a mistake in the notes in the definition of an fpqc cover.

The correct definition is

Def: a covering $(\mathcal{U}_i \to \mathcal{U})_i$ is a fpqc covering if

- each $\mathcal{U}_i \to \mathcal{U}$ is flat
- the induced map $\sqcup \mathcal{U}_i \to \mathcal{U}$
- ~~For each $R \in k\text{-Alg}$, $\exists$ an fpqc base change $R \to R'$ s.t. $\mathcal{U}(R')$ is the union of images of all $\mathcal{U}_i(R')$~~.

Remark: For the fpqc topology this last condition (surj on points after fpqc base change) is equiv to being jointly surjective:

$\Rightarrow$) This is the easy direction: Any $u \in \mathcal{U}$ defines a morphism $\mathrm{Spec}(k(u)) \to \mathcal{U}$. Apply the surj on points after fpqc base change condition to $R = \mathrm{Spec}(k(u))$ and deduce the joint surj condition.

$\Leftarrow$) Let $\mathcal{U} = \mathrm{Spec}(A)$, $\mathcal{U}_i = \mathrm{Spec}(A_i)$ given a family of fpqc morphisms $\{\mathcal{U}_i \to \mathcal{U}\}$ and using that affine schemes

that are jointly surj,
are quasi-compact, one can
pullback all the morphisms $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$
and obtain another fpqc family

$$\{ \mathrm{Spec}(\underbrace{A_i \otimes_A R}_{=: B_i}) \to \mathrm{Spec}(R) \}$$

The $\mathcal{U}_i \to \mathcal{U}$'s being jointly surj, so are the
$\mathrm{Spec}(B_i) \to \mathrm{Spec}(R)$'s, so $\forall\, \mathfrak{p} \subseteq \mathrm{Spec}(R)$
$\exists\, i$ and $q_i \in \mathrm{Spec}(B_i)$ such that $q_i \cap R = \mathfrak{p}$
which translates geometrically as:

Δ This is where fpqc (or fppf) is required

show that $\bigcup_i \mathrm{im}(\mathrm{Spec}(B_i) \to \mathrm{Spec}(R)) = \mathrm{Spec}\, R$

this fails if you replace faithful flatness by étale

Set $R' = \prod_i B_i$. By what preceeds the map $R \to R'$
is fpqc. Let's evaluate $\bigcup_i \mathcal{U}_i \to \mathcal{U}$:

- an element of $\mathcal{U}(R')$ is a ring map

$$A \to \prod_i B_i$$

- composing with the projection $\prod_i B_i \to B_i$
gives an element of $\mathcal{U}_i(R')$ so one gets

$$\mathcal{U}(R') = \bigcup_i \mathrm{im}(\mathcal{U}_i(R') \to \mathcal{U}(R'))$$