Homework set 4

HW1. Let X be a simply connected space and let $n \geq 2$. There is a natural homomorphism

$$\Omega\colon H^n(X) \to H^{n-1}(\Omega X)$$

defined by looping $x\colon X \to K(\mathbb{Z}, n)$ and identifying $\Omega K(\mathbb{Z}, n)$ with $K(\mathbb{Z}, n-1)$.

Show that the Serre spectral sequence of the path-loop fibration $\Omega X \to PX \to X$ satisfies

(1) $E_n^{0,n-1} = im(\Omega) \subseteq H^{n-1}(\Omega X)$, and
(2) $d_n(\Omega x) \in E_n^{n,0}$ is equal to the image of x under the surjection $H^n(X) \to E_n^{n,0}$.

This shows that the transgression $\tau = d_n\colon E_n^{0,n-1} \to E_n^{n,0}$ is 'inverse' to Ω in a certain sense.

HW2. Let B be a simply connected space and let $n \geq 1$. Let $x \in H^{n+1}(B)$ and consider the associated principal fibration

$$K(\mathbb{Z}, n) \to E \to B.$$

Show that the E_{n+1}-page of the Serre spectral sequence with $\mathbb{Q}$-coefficients is given by

$$E_{n+1}^{p,q} \cong H^p(B; \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n); \mathbb{Q}),$$

with differential determined by

$$d_{n+1}(1 \otimes a) = x \otimes 1,$$

where $a \in H^n(K(\mathbb{Z}, n); \mathbb{Q})$ is the canonical generator.

(As discussed in class, a hint for both problems is to compare with a suitable path-loop fibration over an Eilenberg–Mac Lane space and use naturality of the Serre spectral sequence.)

Deadline: 2025–11–20. If you have used any resources outside the course literature/lecture notes, please indicate this in your solution. Similarly if you have discussed the problems with another student, or used AI. Hand in your solutions by e-mail to alexb@math.su.se