ALGEBRAIC GROUPS

PROBLEM SHEET 4

Please hand in your solutions to the following problems by 2026-05-22, either by email or in my mailing box at the department or during the lectures. You are encouraged to discuss the problems with your colleagues, but must hand in your own solutions. You are welcome to ask questions.

Problem 1. Let $\mathbf{G}$ be a linear algebraic group over an algebraically closed field k, denote by $\mathbf{R_u}(\mathbf{G})_k$ its unipotent radical. Show that for any algebraically closed field extension K of k the following equality $\mathbf{R_u}(\mathbf{G})_K = \mathbf{R_u}(\mathbf{G}_K)$ is satisfied.

Problem 2. Let k be an imperfect field of characteristic $p > 0$ and let k' be a purely inseparable field extension of degree $p^n > 1$. Let $\mathbf{G} := \prod_{k'/k} \mathbf{GL}_1$ be the Weil restriction of $\mathbf{GL}_1$ from k' to k. This is a smooth k-group scheme of dimension p^n. Show that $\mathbf{G}$ is not reductive.

Hint: Show that $\mathbf{GL}_1$ seen as a k-group scheme is a k-subgroup scheme of $\mathbf{G}$. Then show that the smooth connected quotient $\mathbf{G}/\mathbf{GL}_1$ is unipotent.[1]

Problem 3. [2] Let k be an algebraically closed field and $\mathbf{G} \subset \mathbf{GL}_3$ be the connected solvable subgroup defined, for any k-algebra R, by:

$$\mathbf{G}(R) = \left\{ \begin{pmatrix} t & x & z \\ 0 & 1 & y \\ 0 & 0 & t^{-1} \end{pmatrix} : t \in R^\times,\ x, y, z \in R \right\}.$$

(1) Show that $\mathbf{G}$ is not reductive,
(2) Show that the torus defined by $\mathbf{T}(R) := \{\mathrm{diag}(t, 1, t^{-1}) \mid t \in R^\times\}$ is a maximal torus of $\mathbf{G}$ and compute the weight space decomposition for $\mathfrak{g}$ under the adjoint action of $\mathbf{T}$.
(3) Show that the nontrivial character $\chi(t) = t$ occurs with multiplicity 2, while its inverse $\chi(t) = t^{-1}$ occurs with multiplicity 0.

Problem 4. Let k be a unital commutative ring and $\mathbf{G} \to \mathrm{Spec}(k)$ be a reductive group scheme[3] and let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. Remember that for any cocharacter λ, one can define a subfunctor $\mathbf{P_G}(\lambda) \subset \mathbf{G}$ by:

$$\mathbf{P_G}(\lambda)(R) := \{g \in \mathbf{G}(R) \mid \lim_{t \to 0} \lambda(t) \cdot g \text{ exists}\} \text{ for any } k\text{-algebra } R.$$

Show that $\mathbf{P_G}(\lambda) = \mathbf{P_G}(\lambda^n)$ for any $n > 0$.

Problem 5. Let k be an algebraically closed field, and let $\mathbf{G} \to \mathrm{Spec}(k)$ be a reductive group scheme[4] and $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. Any cocharacter $\lambda : \mathbf{G}_m \to \mathbf{T}$, determines an action of $\mathbf{G}_m$ on $\mathfrak{g} := \mathbf{Lie}(\mathbf{G})$. Let $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ be the corresponding weight space decomposition. Show that

$$\mathrm{Lie}(\mathbf{P_G}(\lambda)) = \bigoplus_{n \geq 0} \mathfrak{g}_n,\ \mathrm{Lie}(\mathbf{U_G}(\lambda)) = \bigoplus_{n > 0} \mathfrak{g}_n,\ \mathrm{Lie}(\mathbf{Z_G}(\lambda)) = \mathfrak{g}_0,$$

where, as a reminder $\mathbf{U_G}(\lambda) \subset \mathbf{G}$ is defined by

$$\mathbf{U_G}(\lambda)(R) := \{g \in \mathbf{G}(R) \mid \lim_{t \to 0} \lambda(t) \cdot g = 1_{\mathbf{G}}\} \text{ for any } k\text{-algebra } R,$$

[1] You don't need to show that this quotient is representable, nor that it is smooth and connected. For those who want to know more about quotients of groups of finite type over fields, a nice summary is available here (see in particular Theorems 5.2.5 and 5.2.9), see also this MathOverflow post.

[2] In the lectures we have seen that when $\mathbf{G}$ is a reductive group scheme acted on by a maximal torus, the root spaces are 1-dimensional. The aim of this exercise is to show that the reductive assumption cannot be removed here.

[3] Actually this is enough to consider that $\mathbf{G}$ is an affine finitely presented group scheme over k.

[4] What follows also work étale-locally when k is a commutative unital ring, but we focus on the algebraically closed field to avoid technicalities.

and $\mathbf{Z}_{\mathbf{G}}(\lambda)$ is such that $\mathbf{Z}_{\mathbf{G}}(R)$ is the subset of $\mathbf{G}(R)$ whose points commute with the action induced by λ, for any k-algebra R.

Problem 6. Let $\mathbf{G}$ be $\mathbf{SL}_n$ or $\mathbf{PGL}_n$ over a commutative unital ring k and let $\mathbf{T}$ be the diagonal torus. In both cases show that $\mathbf{Z}_G(\mathbf{T}) = \mathbf{T}$ (so $\mathbf{T}$ is a maximal torus on all geometric fibres) by using the Lie algebras.