Lecture 13. Generalities on reductive groups. ①

Reminder:

• Let S be an (affine) scheme. A $\left\{\begin{array}{l}\text{semi-simple}\\ \text{reductive}\end{array}\right.$ S-group is a smooth S-affine gp scheme $G \to S$ s.t. the geometric fibres $G_{\bar s}$ are connected reductive / groups. {semi-simple gps}

Geometric fibers are only well-defined up to non canonical isomorphism over s as they correspond to the choice of an algebraic closure Ω of $k(s)$ (together with a map $k(s) \to \Omega$).

⇝ a priori one should check reductivity of G_Ω for any such Ω, but actually it is enough to do it for one geom. point over s as $\forall$ linear alg gp H over an alg closed field k, and any alg gp ext $k \to K$, $R_u(H)_K = R_u(H_K)$.

Actually:

* unipotence makes sense over any field k: a linear alg gp G over a field k (not necessarily algebraically closed) is unipotent if $\exists$ an alg closed extension $\bar{k}$ of k s.t. $G_{\bar k}$ is unipotent.

This works for linear alg. gps. If G is not of finite type one still has a no° of unipotence over any field (see DG, IV.2, Prop 2.5 or SGA 3, VIIb, 2.8)

②

* From this, one can define:

Def: Let G be a smooth affine k-gp. The k-unip radical $R_{u,k}(G)$ of G is the maximal smooth connected unip. normal k-subgp of G.

Let G be a smooth connected affine k-gp and K/k be a field extension.

In general one only has $R_{u,k}(G)_K \subseteq R_{u,K}(G_K)$
BUT if K/k is separable the above inclusion is an equality.

(see Conrad-Gabber-Prasad chapter 1 for a general treatment AND the analogous introduc of $R_k(G)$).

• We ended last lecture by claiming that the above def of reductivity is reasonable because it imposes the minimal conditions to ensure that reductivity of a fibre is inherited by nearby fibers:

Conrad 3.1.9

Prop: Let $G \to S$ be a smooth affine gp scheme$/S$ and suppose $G^\circ_{\bar s}$ is reductive for some $s \in S$

① $\exists\, \mathcal{U} \ni s$ open neighborhood s.t. $G^\circ_{\bar u}$ is reductive $\forall u \in \mathcal{U}$. The same holds for semi-simplicity

② If $T \subseteq G$ is a torus s.t. $T_{\bar s}$ is max in $G^\circ_{\bar s}$

for some $s \in S$, then $\exists$ an open $\mathcal{V}$ around $s \in S$ ③

s.t $Z_G(T)^0_{\mathcal{V}} = T_{\mathcal{V}}$ and $T_{\bar{s}}$ is a max torus in $G^0_{\bar{s}}$ $\forall s \in \mathcal{V}$.

Remarks: ① The connected component (at identity) is def. in all generalities in Pb sheet 3, exercise 8. But as here we are only interested in geom pts over s, the "classical" def is enough.

② **Reminder:** T is a torus if $\exists$ an fppf cover $\{S_i\}$ of S st $G_{S_i} \cong \mathcal{D}(\mathbb{Z}^{n_i})_{S_i}$ for each i:

Have $A_i[\mathbb{Z}^{n_i}]$

where $S_i = \mathrm{Spec}(A_i)$ and $A[\mathbb{Z}^{n_i}] =$ gp alg of $\mathbb{Z}^{n_i}$.

③ $Z_G(T)$ is def as in lecture 3 as the scheme theoretic centralizer of T. It is a smooth closed S-subgp of G (left as an exercise – This uses the smoothness criterion with square free ideal).

④ Statement of ② in the above prop assume that T exists. It always does étale locally for G reductive as we will see later in the lecture.

④

⑤ The above Proposi$^=$ is SGA 3, XIX 2.6 without assuming that G_s is connected $\forall s \in S$. The key point behind this is

Prop 3.1.12 Conrad:
Let $G \to S$ be a smooth S-affine gp s.t each $G^0_{\bar{s}}$ is reductive. The locus of $s \in S$ s.t G_s is connected is close in S. In part if S is irred. & the generic fibre is connected then all fibers are connected

fibre at the generic pt

in the affine case:

$S = \mathrm{Spec}(A)$ is irr $\Leftrightarrow$ A has exactly one min prime ideal

unique point contained in every non empty open subscheme

this def the generic pt.

⑥ Before explaining some key points of the proof of the above Proposition, let us mention an important consequence: reductivity and semi-simplicity can be descended. More specifically:

Prop: Let • $\{A_i\}$ be a directed system of rings ⑤
s.t. $\varinjlim A_i = A$
• G_{i_0} a smooth affine A_{i_0}-gp for some i_0.

For all $i \geq i_0$, set $G_i := G_{i_0} \otimes_{A_{i_0}} A_i$
and $G = G_{i_0} \otimes_{A_{i_0}} A$

Then the fibres of $G^\circ \to \mathrm{Spec}(A)$ are reductive if and only if the fibres $G_i^\circ \to \mathrm{Spec}(A_i)$ are reductive & sufficiently large $i \geq i_0$
and G is a reductive A-gp iff G_i is a red A_i-gp & sufficiently large $i \geq i_0$

Same for semi-simplicity.

? Why is it a corollary & why is it useful?

For $i \geq i_0$, let $\mathcal{U}_i \subset \mathrm{Spec}(A_i)$ be the locus of points at which the geometric fibre of G_i° is reductive (resp semi-simple), and def $\mathcal{U} \subset \mathrm{Spec}(A)$ similarly for G. ① of the above prop tells us that the $\mathcal{U}_i$'s & $\mathcal{U}$ are open.

This is crucial to show that $\mathcal{U} = \mathrm{Spec}(A)$ iff $\mathcal{U}_i = \mathrm{Spec}(A_i)$
Then, one shows that if $G^\circ = G$ then $G_i^\circ = G$ & sufficiently large i.

This uses EGA IV_3, §8.7 ii) but this goes beyond what I want to cover here. ⑥

Why is it useful? Because it allows one to reduce many pb to the noetherian case: Most prop one wants to show are Zariski local so one can reduce to S affine (affine opens being a basis for the Zariski topology). But then in part S is a direct limit of noetherian rings (EGA IV §8).

Key arguments to prove the proposi^{-}:

* Start by proving ② (you need it for ①):

This requires the following (VERY IMPORTANT) fibral isomorphism criterion (lemma B.3.1 in Conrad's notes).

Lemma: Let $h: Y \to Y'$ be a map between locally finitely presented schemes over a scheme S. Assume that Y is S-flat.
Then, if h_s is an iso $\forall s \in S$ so is h.

How does it apply here?

$$T \subset Z_G(T)^\circ \subset Z_G(T)$$

T is S-flat as (*)

$Z_G(T)$: smooth closed S-subgp scheme of G $\Rightarrow$ loc of finite pres as G is loc of f.t.

(*) = Remind that an equiv def of tori is: gps of mult type (in part loc. of finite type) with smooth connected fibres

Now any s-gp scheme loc. of finite type (7)
with smooth geom. fibers is flat over S
Work Zariski locally around s_0
Apply B.3.1 to $T = Y$ & $Z_G(H)^\circ = Y'$: one only
needs to check that $\forall \xi \in S$, $T_\xi = Z_G(T_\xi)^\circ$

This is done by dimension arg (because we work Zariski locally we may (& must!) assume that dimensions are cst on fibres.

Then apply classical theory to show that
T_ξ is max in G°_ξ

Proof of (5): • Assume $G^\circ_s \neq 1$ and fix a geom pt $\bar{s}$ over S
$= \mathrm{Spec}(\overline{k(s)})$

Use: Any max torus in $G^\circ_{\bar{s}}$ descends to a split torus in G°_K for some finite extension $k(s) \hookrightarrow K$.

Why? * fppf/Galois descent for finite type subgp schemes: $I = \mathrm{Gal}(\overline{k(s)}/k(s))$
$\forall \sigma \in I$ induces an automorphism
$$\sigma^*: G_{\overline{k(s)}} \longrightarrow G_{\overline{k(s)}}$$

(8)
Applying this to $T_{\overline{k(s)}}$ one gets another torus $\sigma^*(T_{\overline{k(s)}}) \subset G_{\overline{k(s)}}$

Let $\mathrm{Stab}_I(T_{\overline{k(s)}}) := \{\sigma \in I \mid \sigma(T_{\overline{k(s)}}) = T_{\overline{k(s)}}\}$

Fact $[I : \mathrm{Stab}_I(T_{\overline{k(s)}})] < \infty$
because $T_{\overline{k(s)}}$ is smooth, closed, of finite type.

This corresponds to a finite extension K of $k(s)$
and $T_{\overline{k(s)}}$ descends to $T_K \subset G_K$.

* Any torus $T_{\overline{k(s)}}$ becomes split after a FINITE separable extension of $k(s)$ $\Rightarrow$ The splitting field is the finite ext over which I acts triv. on $X^*(T) = \mathrm{Hom}_{\text{gp sch}/k(s)}(T, \mathbb{G}_m)$

Last time we saw this can be checked on k_{sep}. This is due to the fact that $(k(s))_{sep} = \varinjlim_{\substack{K \hookrightarrow k(s) \\ \text{sep + finite}}} K$

and that being split is equiv to require $I \curvearrowright X^*(T_{\overline{k(s)}})$ trivially which already happen after a finite extension as $X^*(T)$ is f.g. and I acts trivially if it acts trivially on generators

ms Waterhouse 7.3.

and, one can work fppf loc

(split tori are fppf loc. iso to $\mathbb{G}_m^{\oplus n}$)

split tori of red gps!

$\rightsquigarrow$ Take an fppf neighborhood of s so that one can extend $k(s)$ to K.

* Use:

Proposition (3.1.6 Conrad): Let $G \to S$ be a smooth S-affine gp scheme and H_0 be a mult type subgp of G_s over some $s \in S$.

There exists an étale neighborhood (S', s') of (S, s) with $k(s') = k(s)$ and a mult type subgp $H' \subset G_{S'}$ s.t. $H'_{s'} = H_0$ $(G_{S'})_{s'} = G_s$

closed immersion (see last lecture)

To make an additional étale base change around s so that T_s lifts locally to a split torus T of G

Then apply ②: Zariski-locally around s

$Z_G^0(T) = T \simeq \mathcal{D}_S(\mathfrak{M}) = \mathrm{Hom}_{S\text{-}gp}(\mathfrak{M}_S, \mathbb{G}_m)$

finite free ab gp.

Remaining dust:

use the ac of T on $\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(G^0)/_S$ and the following lemma:

Lemma (3.1.10. Conrad)

Let G be a smooth connected affine group scheme over an alg closed field k and T be a max torus in G s.t $Z_G(T) = T$.

For each non zero T-weight α on $\mathfrak{g}$, let T_α be the codim-1 subtorus $(\ker \alpha)^0_{red}$

The gp G is reductive iff the smooth connected subgp $Z_G(T_\alpha)$ is red for each α.

In other words reductivity is, once again, governed by the rk 1 case.

? What's going on here ?

$T \curvearrowright \mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Lie}(G^0)$ over S (which is now an appropriate open affine subspace of the "initial" S) & contains s.

The ac decomposes into a direct sum of quasi-coherent weight spaces $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{a \in \mathfrak{M}} \mathfrak{g}_a$

q.c subsheaves of $\mathfrak{g}$ where $t \in S(R)$, $t \cdot v = a(t) \cdot v$

$\mathfrak{g}$ is finite locally free (hence flat) over S (11)

$W_{\mathrm{wt}\,\mathfrak{g}}$ " and $\mathfrak{g}_a$ and $\mathfrak{g}_0$ are direct summands of $\mathfrak{g}$ (hence loc free hence flat)

$\Rightarrow$ the forma° of these weight spaces commute under base change on S.

In short: the $\mathfrak{g}_0$ and $\mathfrak{g}_a$'s are v.b so the weight space decompos° on the s-fibre encode the weight spaces on the nearby fibres (use Nakayama at s)

"if the images of sec° generate the fibre at s then they already generate $\mathfrak{g}$ over an open neighborhood of s."

$\Rightarrow$ By shrinking around s one may assume that all weight spaces have cst rank so $\mathfrak{g}_a \neq 0 \Leftrightarrow a = 0$ or $a \in \Phi(G^0_{\bar{s}}, T_{\bar{s}})$

Now: $\cdot\ \mathfrak{t} \subset \mathfrak{g}_0$ + rank arg $\Rightarrow \mathfrak{t} = \mathfrak{g}_0$

$\cdot\ \mathfrak{g}_a$ is a line bundle

(12)

$T_a = D(M/L)$ is the ring codim 1 subtorus contained in ker a for $T = D(M)$

ensures that M/L is torsion free

$L \subset M$ is the satura° of $\mathbb{Z}a$ in M
$= \{m \in M \mid \exists n \geq 1 \text{ with } nm \in \mathbb{Z}a\}$

add all elem° of M that become int. mult. of elem° of A

$\cdot\ \Phi$ is reduced $\Rightarrow$ elem° in Φ apart from $\pm\alpha$ are lin ind. from α so cannot vanish on any fiber of T_α

Now $Z_{G^0}(T_\alpha)$ is smooth
$\Rightarrow \mathrm{Lie}(Z_{G^0}(T_\alpha)) \subset \mathfrak{g}^{T_\alpha} := \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$
is actually an equality.
and we can replace G by $Z_G(T_\alpha)$ thanks to the lemma.
So we are reduced to the case $\Phi = \{\alpha, -\alpha\}$