Lecture 16 (1)

Reminder: Let $\begin{matrix} G \\ | \\ S \end{matrix}$ be a reductive group scheme

Assume that G admits a split maximal torus T over S and fix an isomorphism $T \simeq D_S(M)$ where M is a finite free $\mathbb{Z}$-module.

def: A **root** for (G,T) is a non zero element $a \in M$ such that $\mathfrak{g}_a$ is a line bundle, called a **root space** for (G,T,M).

Given a root $a \in M$, $T \circlearrowright W(\mathfrak{g}_a)$ via $(t,v) \mapsto a(t)v$ using the vector bundle structure on $\mathfrak{g}_a$.

• Root groups are the image of $W(\mathfrak{g}_a)$ under the unique S-group homomorphism $\exp_a : W(\mathfrak{g}_a) \longrightarrow G$, inducing the canonical inclusion $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$ on Lie alg and intertwining the T-action on G via conjugation and the T action on $W(\mathfrak{g}_a)$ via scaling.

Last time we showed that: to prove existence & uniqueness of $\exp_a$ it is necessary and sufficient to prove existence and uniqueness of a $\mathbb{G}_m$-equiv iso of S-groups $W(\mathfrak{g}_a) \simeq U_G(\lambda)$ that induces the identity map on Lie algebras. (2)

Note: λ was chosen accordingly to $a : T \to \mathbb{G}_m$ in such a way that the induced pairing is strictly positive (so that $U_G(\lambda) = W(\mathfrak{g}_a)$).

Under this identification $T = Z_G(\lambda)$ and $\mathbb{G}_m$ acts on $\mathfrak{g}_a$ via

$$t.v := \underbrace{(a \circ \lambda(t))}_{=:\, t^{\langle a,\lambda\rangle}}.v$$

the subfunctor of points of G that commute with the $\mathbb{G}_m$-action λ.

Uniqueness of a morphism $W(\mathfrak{g}_a) \simeq U_G(\lambda)$?

Comes from the fact that $W(\mathfrak{g}_a)$ has no non trivial $\mathbb{G}_m$-equivariant automorphism that induces the identity on the Lie algebras.

Indeed, working Zariski locally, $\mathfrak{g}_a$ admits a trivialisation as a line bundle so that the above assertion is the same as saying that $\mathbb{G}_a$ over S admits no non trivial automorphism that is $\mathbb{G}_m$-equivariant for the action

$$\mathbb{G}_m \times \mathbb{G}_a \longrightarrow \mathbb{G}_a$$
$$(t, x) \longmapsto t^{\langle a,\lambda\rangle}x$$

Now: • an endomorphism of $\mathbb{G}_a$ over a ring k is precisely an additive polynomial.

• equivariance for $t.x = t^n x$, $n \in \mathbb{Z}_{>0}$

$\Rightarrow$ the polynomial is a monome of deg. 1, hence reads as scaling on the Lie alg (say by c)

• Inducing identity on Lie alg $\Rightarrow c = 1$.

Existence of a $\mathbb{G}_m$-equiv isomorphism $W(\mathfrak{g}_a) \simeq \mathcal{U}_G(\lambda)$

③ • A first observation: $\mathcal{U}_G(\lambda)$ has connected unipotent fibres of dimension 1

It is smooth with the algebra $\mathfrak{g}_a$ (which is a line bundle)

Conrad Thm 4.1.7 (2) and (4)

$\Rightarrow$ The geometric fibres are $\mathbb{G}_a$ (DG, IV §2 Cor 2.9)

Remember that over an alg. closed field unip gps are $\mathbb{G}_a$ in char 0 or $\mathbb{G}_a$, $\mathbb{Z}/p\mathbb{Z}$ or μ_p in char $p>0$.

⚠ It is true but not easy to show directly that the actual fibres of $\mathcal{U}_G(\lambda)$ over S are iso to $\mathbb{G}_a$ (due to the fact that the aut. functor of $\mathbb{G}_a$ behaves poorly in char $p>0$). This can be settled as follows:

① Work étale locally (as allowed by the uniqueness of $\exp_a$ (if it exists)). Remember that $\mathcal{U}_G(\lambda)$ is smooth $\Rightarrow$ $\mathcal{U}_G(\lambda) \to S$ is smooth + surjective $\Rightarrow$ $\mathcal{U}_G(\lambda) \to S$ admits a section σ disjoint from the identity section.

because we work étale locally

④ **Idea:** By def of $\mathcal{U}_G(\lambda)$ the orbit map of σ $\mathbb{G}_m \to \mathcal{U}_G(\lambda)$ extends to an S-scheme map

$$q: \mathbb{A}^1_S \longrightarrow \mathcal{U}_G(\lambda)$$

that carries 1 to σ

• is $\mathbb{G}_m$ equiv. where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by scaling (because it is enough to check such equiv. on the open $\mathbb{G}_m \subsetneq \mathbb{A}^1_S$)

• $q(0)=1$ by def of $\mathcal{U}_G(\lambda)$

• $\forall s \in S$, $\mathcal{U}_G(\lambda)_{\bar{s}} \simeq \mathbb{G}_a$ and $\mathbb{G}_m$-action is $\sigma(\bar{s}) \longmapsto 1$

then scaling by t^n, $n = \langle a, \lambda\rangle \in \mathbb{Z}$

by the condi on Lie alg

$\overset{(i)}{\Rightarrow}$ $q_{\bar{s}}$ identifies with an aut. of $\mathbb{G}_a$ over $k(\bar{s})$ s.t $q_{\bar{s}}(t) = t^{\langle a,\lambda\rangle}$ for $t \in \mathbb{A}^1_S$.

And$^{(ii)}$ μ_n acts on $\mathcal{U}_G(\lambda)$ trivially $\Leftrightarrow$ to show $\mathcal{U}_G(\lambda)^{\mu_n} = \mathcal{U}_G(\lambda)$, which is to be checked on the fibres because $\mathcal{U}_G(\lambda)^{\mu_n}$ is smooth (as μ_n is of mult. type). This is ensured by what precedes.

• It remains to show (using (ii)) that $\mathcal{U}_G(\lambda) \simeq \mathbb{G}_a$ over S via an S iso carrying the $\mathbb{G}_m$-ac over to $t.x = t^{\langle a,\lambda\rangle} x$.

Indeed, if such an iso exists it identifies ⑤
$W(y_a)$ with G_a carrying the G_m-ac° over to $t.x = t^{\langle a,\lambda\rangle}x$
so that $W(y_a)$ & $\mathcal{U}_G(\lambda)$ are G_m-equiv isomorphic
as S-gps

How do we get $\exp_a$ from this iso?

any such iso affects y_a
by scaling by a global unit

so scaling reversely on $W(y_a)$ by the reciprocal of that unit provides the expected $\exp_a$.

It remains to show $\mathcal{U}_G(\lambda) \cong G_a$ as G_m-gp schemes over S. We use the S-morphism q:

- G_m-equiv of $q \Rightarrow q$ factors through an S-scheme map

induced by the orbit map
$G_m \to \mathcal{U}_G(\lambda)$
$t \mapsto t.\sigma$

$$\begin{tikzcd} A^1_S \arrow[r, "q"] \arrow[d, "t \mapsto t^n"'] & \mathcal{U}_G(\lambda) \\ A^1_S \arrow[ur, dotted, "\bar q"'] & \end{tikzcd}$$

that carries 0 to 1 and that is G_m-equiv for the action $t.x = t^n x$ on the affine line & the conjug° via λ on $\mathcal{U}_G(\lambda)$.

Using our previous work on $q_{\bar s}$ $\forall s \in S$ one shows that on geom fibres $\bar q_{\bar s}$ identifies with an ⑥
automorphism of $A^1_{k(\bar s)}$ so $\bar q$ is an iso of S-schemes
by the fibral iso criterion.

It remains to prove it is an S-gp homomorphism

$\bar q$ inherits an ac° by the quotient $G_m/\mu_n \cong G_m$
(check it! It is not hard but you need to justify it) that makes the S-scheme iso $\bar q$ identify $\mathcal{U}_G(\lambda)$ with an S-gp structure on A^1_S that has 0 as the identity and is equiv. for the ordinary G_m-scaling

$\Rightarrow$ it remains to show that addition is the only such gp law on A^1_S.

~~us~~ reduce yourself to $S = \mathrm{Spec}(k)$ (a ring) and use homogeneity considerations on t to conclude.

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called a parametrization of $\mathcal{U}_a$

Remark: The existence of $\exp_a$ implies that to give a G_m-equiv isomorphism $p_a : G_a \cong \mathcal{U}_a$ (for resp. the T-ac° on $\mathcal{U}_a$ and the ac° $t.x = a(t)x$ on G_a), amounts to choosing a global trivializing sec° X of the line bundle y_a via $p_a(z) = \exp_a(zX)$
(to get convinced use the faithful flatness of $a : T \to G_m$ and the fact that the only G_m-equiv aut. of G_a are scaling by global units of S.

?) Now that we are given roots and associated root gps how do we define coroots? (7)

Pb: Classical theory uses $\mathcal{D}(Z_G(T_a))$ and the classifica° of split semisimple gps of rank 1.

$:= \ker(a)^{\circ}_{red}$

More specifically, if G is a reductive gp over an alg closed field, T is a maximal torus, $\forall a \in \Phi(G,T)$ there exists a unique, up to $T(k)$-conj, isogeny $\varphi_a : Sl_2 \to G$

- carrying D onto T and $U^{\pm}$ onto $U_{\pm a}$,

diag. matrices — upper triang matrices

- s.t $\ker(\varphi) \subseteq \mu_2$

Using $\varphi_a : Sl_2 \longrightarrow G$ one defines the coroot

$$\begin{array}{ccc} Sl_2 & \longrightarrow & G \\ \cup & & \cup \\ D & \longrightarrow & T \end{array}$$

associated to a by

$$a^\vee : \mathbb{G}_m \longrightarrow D \longrightarrow T$$

$$c \longmapsto \begin{pmatrix} c & \\ & 1/c \end{pmatrix}$$

$$\begin{pmatrix} c & \\ & 1/c \end{pmatrix} \longmapsto \varphi_a \begin{pmatrix} c & \\ & 1/c \end{pmatrix}$$

BUT: We haven't def the derived gp in such generalities & we don't have yet a classifica° of semi-simple rank 1 gp. (8)

Key idea: Simultaneously characterize the coroot $a^\vee : \mathbb{G}_m \longrightarrow G$ (we will replace here G by $Z_G(T_a)$ to reduce ourselves to $\mathfrak{g} = \mathfrak{t}_a \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$) and compatible trivializa° of $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$

This requires to make use of the "open cell", $\Omega_a = \exp_a(W(\mathfrak{g}_a) \times T \times W(\mathfrak{g}_a)) \subseteq Z_G(T_a)$

Remember from last time that $\exp_{\pm a}$ is a closed immersion factoring through $Z_G(T_a)$, and the multiplica° map

$$W(\mathfrak{g}_{-a}) \times T \times W(\mathfrak{g}_a) \longrightarrow Z_G(T_a)$$

$$(x', t, x) \longmapsto \exp_{-a}(x')\, t\, \exp_a(x)$$

is an iso onto $\Omega_a \subset Z_G(T_a)$.

Theorem (Conrad 4.2.6): Let G be a reductive gp scheme over S with fibres of semi simple rank 1. Assume $\exists$ a split max torus $T = D_S(M) \subset G$ and a root $a : T \longrightarrow \mathbb{G}_m$.
There is a unique pair $(\beta_a, a^\vee)$ consisting of an $\mathcal{O}_S$-bilinear (hence $\mathbb{G}_m$ equiv) pairing of line

bundles $\beta_a : \mathfrak{g}_a \times \mathfrak{g}_{-a} \longrightarrow \mathcal{O}_S$

$(x, y) \longmapsto xy$

and an S-homomorphism $a^\vee : \mathbb{G}_m \longrightarrow T$ s.t the following condi° hold:

(i) $\forall S' \in Sch/_S$ and $\forall \exp_a(x) \in \mathcal{U}_a(S')$, $\exp_{-a}(y) \in \mathcal{U}_{-a}(S')$ the S'-valued pt

$$\exp_a(x)\exp_{-a}(y) \in \Omega_{-a} \subset G$$

lies in Ω_a iff $1 + xy$ is a unit on S'

(ii) When this unit condi° is satisfied

$$\underbrace{\exp_a(x)\exp_{-a}(y) = \exp_{-a}\left(\frac{y}{1+xy}\right)a^\vee(1+xy)\exp_a\left(\frac{x}{1+xy}\right)}_{\in\ \Omega_a}$$

In part. the forma° of this bilinear pairing and $a^\vee$ commute with base change on S.

Moreover the pairing $(x, y) \mapsto xy$ is a perfect duality and $a \circ a^\vee = 2$ (namely $a(a^\vee(c)) = c^2$ for $c \in \mathbb{G}_m$)

Remarks: (1) As a consequence, $\mathfrak{g}_a$ is globally trivial if and only if so is $\mathfrak{g}_{-a}$. Likewise $\mathcal{U}_a$ admits a parametriza° iff so does $\mathcal{U}_{-a}$.

When such param. $p_{\pm a} : \mathbb{G}_a \simeq \mathcal{U}_{\pm a}$ exist they are said to be linked if they correspond to dual bases for $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$, called dual trivializa°.

(2) For a given parametriza° of $\mathcal{U}_a$ there is a uniq $\mathcal{U}_{-a}$ to which it is linked.

(3) Necessarily $(-a)^\vee = -a^\vee$ (use classical theory to get the equality on geom fibres then use the fibral iso crit.)

Proof Assume that S is non empty.

(1) To prove uniqueness show that it is harmless to treat 2 cases: $S = Spec(k)$ for k an alg closed field and $Z_G = 1$ over a general S.

This comes from the facts that: one can reduce oneself to $G = G/_{Z_G}$ and that any S-homomorphism $\mathbb{G}_m \rightarrow T$ is uniquely determined by its effect on geom fibres.

(2) Consider the situa° over $k = \bar{k}$ alg closed. so that $G/_{Z_G} = PGl_2$ and show existence and uniqueness of $(\beta_a, a^\vee)$.

You will need to run the Sl_2 case explicitely & to make use of the deg 2 central isogeny $Sl_2 \rightarrow PGl_2$; and to explicit how properties (i)&(ii) required for the pairing transport through $G \rightarrow G/_{Z_G}$

(3) Back to the general case, results over $\bar{k}$ obtained in (2) imply uniqueness of $a^\vee$ and perfectness of the bilinear pairing (if it exists) and the identity $a \circ a^\vee = 2$ (once again to be showed on geom fibres)

Existence & uniqueness of β_a and $a^\vee$ are then ensured via the identifica° $\mu : \mathbb{G}_m \simeq D$, $t \mapsto \begin{pmatrix} t & \\ & 1/t \end{pmatrix}$

when $Z_G = 1$ by:

Prop 4.2.7 (Conrad): Let $G \to S$ be a reductive with trivial centre and geom fibres of semisimple rank 1. If $\exists$ a split max torus $T \subset G$ then Zariski locally on S, $\exists$ a gp isomorphism $G \simeq PGl_2$, $T \mapsto D$.

④ Finally if $Z_G \neq 1$, work Zariski locally on S (allowed by uniqueness of $(\beta_a, a^\vee)$) and arrange, using Prop 4.2.7, that

$$(G/_{Z_G}, T/_{Z_G}) \cong (PGl_2, D)$$

The result follows by studying the following pullback diag

$$\begin{tikzcd}
1 \arrow[r] & Z_G \arrow[r] \arrow[d, equal] & \tilde{G} \arrow[r] \arrow[d, "f"] & Sl_2 \arrow[r] \arrow[d] \arrow[l, dashed, bend right] & 1 \\
1 \arrow[r] & Z_G \arrow[r] & G \arrow[r] & PGl_2 \arrow[r] & 1
\end{tikzcd}$$

$f^{-1}(T) \to T$

central extension

and show that the top row splits (Prop 4.3.1 Conrad)

so that $\tilde{G} \cong Sl_2 \times Z_G$, $\tilde{T} \mapsto D \times Z_G$

Thus we get a cocharacter $a^\vee : \mathbb{G}_m \to T$, $\mathbb{G}_m = D$

? Local splitness vs global splitness

Def: Let $G \to S \neq \emptyset$ reductive gp. G is split if $\exists$ a max torus T equipped with an iso $T \simeq D_S(M)$ for a finite free $\mathbb{Z}$-module M s.t.

① The non trivial weights $a : T \to \mathbb{G}_m$ that occur on $\mathfrak{g}$ arise from elem^t of M (in part. such a are roots for (G, T) and are constant sec° of M_S)

② each root space $\mathfrak{g}_a$ is free of rank 1 over $\mathcal{O}_S$

③ each coroot $a^\vee : \mathbb{G}_m \to T$ arises from an element of $M^\vee$

Remark: When S is connected only condi° ② is not automatic, but here one often need to work with disconnected bases as localiza° on the base and esp. descent theory leads to disconnected base schemes.

Lemma: Any reductive gp scheme $/_{S \neq \emptyset}$ becomes split étale locally on the base

Thm: Let (G, T, M) be a split reductive gp with geom fibres of semi-simple rank 1 over $S \neq \emptyset$. Up to forming a direct pduct against a split central torus, Zariski locally on S, the pair (G, T) is iso to (Sl_2, D) or $(PGl_2, \bar{D})$ or $(Gl_2, \tilde{D})$