Lecture 14

Maximal tori:

(?) These can be def by several ways for smooth affine gp schemes, these ≠ def coincide at least Zariski locally for red. gps.

Def: A max torus in a smooth S-affine grp scheme $G \to S$ is a torus $T \subset G$ s.t. $\forall$ geom. point $\bar{s}$ of S the fibre $T_{\bar{s}}$ is not contained in any strictly larger torus in $G_{\bar{s}}$.

Remarks: (1) $\forall s \in S$ one only needs to consider a single geom pt over s.

This can be shown using Conrad's Prop 3.2.2 (very good illustra° of the spreading out & 'specializa°' pciple.)

you want to show that if K/k is any extension of alg closed fields, and $T \subset G$ is a torus in a smooth affine k-gp then T is not contained in a strictly larger torus of G_K

⇐] is clear by descent.

⇒] Spreading out: K is a direct limit of its k-subalg A_i of finite type over k + limit/descent argument to get that any strict containm^T $T_K \underset{\neq}{\subset} T'$ between K-tori in G_K descends to $T_{A_i} \underset{\neq}{\subset} T'_{A_i}$ between A_i tori in G_{A_i} for some i.

specializa°: $G_{A_i} = G \times_k \operatorname{Spec}(A_i)$ reduced$/k$

$$\begin{array}{c} G_{A_i} \\ \cup \\ T_{A_i} \underset{\neq}{\subset} T'_i \end{array}$$

both reduced $/k$

proper closed subset

& $k = \bar{k}$

$\Rightarrow$ Nullstellensatz applies!

$\Rightarrow \exists$ a k-pt of T'_i that is not in T_{A_i}

this pt lies over some $\xi \in \operatorname{Spec}(A_i)(k)$ leading to a $\lightning$.

(2) How does the above definition relate with the naive (global) def of maximality, namely: an S-torus T of a smooth S-affine gp scheme $G \to S$ is maximal if any containment $T \subset T'$ of S-tori is an equality?

Equality on fibres $\Rightarrow$ global equality by Lemma B.3.1 (fibral iso criterion, seen last time).

Let $h : Y \to Y'$ be a map between loc. finitely presented schemes over a scheme S and assume that Y is S-flat. If h_s is an isomorphism $\forall s \in S$ then h is an iso.

Now consider the converse: if a torus $T \subset G$ is not contained in a strictly larger torus over S then is T max in the sense of the above def?

There are Zariski local obstructions: $T_{\mathcal{U}}$ may lie in a strictly larger $G_{\mathcal{U}}$-torus for some open $\mathcal{U} \subseteq S$.

This translates the fact that the dimension of the max tori in each geom fibre $G_{\bar{s}}$ may not be locally in s.

E.g.: $S = Spec(A)$ with $A = DVR$

and π = uniformizer

t = generic point of S

s = closed point

$$G = Spec(A[X, X^{-1}, u]/(1 - X + \pi u)) \simeq Spec(A[X^{-1}, u]) \to S$$

with $X = 1 + \pi u$ is a smooth affine S-gp with composition law given on $G(R)$ by $(x, u)(x', u') = (xx', \pi u u' + u + u')$

So $G_t \simeq (\mathbb{G}_m)_t$ while $G_s \simeq (\mathbb{G}_a)_s$

Hence $T = 1$ is max relative to containment over S but not over t.

③ Let k be a field maximality for k-tori in the geom. sense is the same as maximality in the k-rational sense of containment of k-tori. This is shown in Conrad's notes A.1.2.

In part. the common dimension of the max k-tori is called the reductive rank of G because it coincides with the same invariant for $G_{\bar{k}} / R_u(G_{\bar{k}})$

④ For reductive group schemes the maximality property is robust wrt to the Zariski topology. For instance ② of the main proposition of last lecture implies that if $T \subset G$ is a torus in a reductive gp scheme $G \to S$ and it is max on the $\bar{s}$-fibre for some $s \in S$, then it is max in the sense of the above def over a Zariski open neighbourhood of s.

Theorem (3.2.6): Let $G \to S$ be a smooth S-aff gp scheme s.t the identity component $G^0_{\bar{s}}$ of each geom fibre, the max tori are their own centralizers. Then the functor on S-schemes

$$Tor_{G/S} : S' \mapsto \{\text{maximal tori in } G_{S'}\}$$

is represented by a smooth quasi-affine S-scheme $Tor_{G/S}$ and $Tor_{G/S} \to S$ is surj.

actually S-affine

see SGA 3, XII, 5.4

If T is a max torus, then the map $G/N_G(T) \to Tor_{G/S}$ def by G-conj against T is an isomorphism. In part. any 2 max tori of G are conjugate étale-locally on S.

Hence

Corollary: Any reductive gp scheme $G \to S$ admits a max torus étale locally on S. In part any connected red gp over a field k admits a geom max torus def over a sep ext of k.

Key arguments of the Thm.

① Work fppf locally as fppf descent is effective for quasi-affine S-schemes. Choose $s \in S$. Main Prop of last lecture $\Rightarrow$ fppf locally around $s \in S$, $\exists\, T \subset G$ s.t. $T_{\bar{s}}$ is max in $G^{\circ}_{\bar{s}}$.

Fibral iso criterion + main prop again we may arrange that $T = Z_G(T)^{\circ}$ (after passing to a Zariski open neighborhood)

$\Rightarrow T$ is max in G

② Show that $G/N_G(T)$ exists as smooth affine S-scheme + surj of $G/N_G(T) \longrightarrow S$ is part of Conrad's Thm 2.3.1 (not showed in class)

③ $G \curvearrowright \mathrm{Tor}_{G/S}$ via conj and $G/N_G(T)$ is a subfunctor of $\mathrm{Tor}_{G/S}$ (to get convinced consider the stabilizer of $T \in \mathrm{Tor}_{G/S}(S)$)

It remains to show that the above containment is an equality: show that this amounts to show that maximal tori $T_{S'}$ and T' in $G_{S'}$ are conjugate fppf-locally on S.

Noetherian reduc: arg in the same spirit as last time allows one to take S noetherian, affine, finite type over $\mathbb{Z}$.

This allows Artin approxima: that leads to the expected conclusion.

Artin approx?

Let R be a local ring that is essentially of finite type over $\mathbb{Z}$, namely $R \cong \underset{\text{mult set}}{S^{-1}}(\mathbb{Z}[x_i]_{i=1}^{n}/$ (ideal of $\mathbb{Z}$

B a finite type R-alg equipped with a map $f: B \longrightarrow \hat{R} := \varprojlim_{n} R/m^N$

Pick $N \geqslant 0$

Let R' be the henselization of R. Then $\exists$ an R-alg map

$$\varphi : B \longrightarrow R'$$

s.t. the induced map to the comple:

$$\hat{\varphi} : B \longrightarrow \hat{R'} = \hat{R}$$

agrees with f modulo $m_{\hat{R}}^{N+1}$.

<u>Proof of Cor</u>: As $Tor_{G/S} \longrightarrow S$ is a smooth surj, it admits sec^n étale locally on S. But section over S-schemes U amount to elem^T in $Tor_{G/S}(U)$, namely max tori in G_U.

Prop 3.6.8: G red $\supset T$ max / S then $W_G(T) := N_G(T) / Z_G(T)$ is finite étale $/S$.

Centers of reductive gps

<u>Def</u>: Let $G \to S$ be a smooth S-affine gp scheme with connected fibres. A reductive centre of G is a central multiplicative type subgp $H \subset G$ s.t. $\forall$ geom pts $\bar{s} \in S$

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\forall"'] & R' \arrow[dl] \arrow[dl, dashed, "\exists!"] \\ R' & \end{tikzcd}$$

R' henselian i.e. any polyn in x having a simple root mod m, has this roots lifting uniquely in A

$H_{\bar{s}}$ is central in $G_{\bar{s}}$ and contains all central multiplicative type subgp $G_{\bar{s}}$.

<u>Remark</u>: In part H is central in G and $\forall$ central mult type subgp of $G_{\bar{s}}$ is contained in $H_{\bar{s}}$.

<u>Thm</u>: Any reductive gp scheme G/S admits a reductive centre Z which coincide with the scheme theoretic centre Z_G of G. In particular Z_G is S-flat. Moreover Z represents the kernel of the action map

$$\mu : G \longrightarrow Aut_S(Tor_{G/S})$$

<u>Proof</u>: work fppf locally on $S \Rightarrow$ assume that G contains a split max torus T $\Rightarrow G/N_G(T)$ identifies with $Tor_{G/S}$ via $g \mapsto gTg^{-1}$

Then construct the red centre & show that it is the scheme theoretic centre as well as the expected centre

Construc: ? Rely on the S: affine case and T split use last lecture to study $T \curvearrowright \mathfrak{g}$ and the dec

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\substack{\mathfrak{g}_a \neq 0 \\ a \in M}} \mathfrak{g}_a \quad \leftarrow \text{finite}$$

working Zariski locally we assume that $\mathfrak{g}_a$ are of constant rank.

Set $\Phi := \{a \in M \mid \mathfrak{g}_a \neq 0\}$ and let $H = \bigcap_{a \in \Phi} \ker(a : T \to \mathbb{G}_m)$

Then apply usual arg to show the expected prop.

Cor: Let G be a reductive S-gp scheme
Z be a mult type subgp scheme of Z_G

The reductive quotient G/Z has center Z_G/Z

In part G/Z_G has trivial centre.

Moreover $\{\text{max tori of } G\} \overset{1:1}{\longleftrightarrow} \{\text{max tori of } G/Z\}$

$$T \mapsto T/Z$$

scheme theoretic preimage under the quotient map. $\longleftarrow$ $G \to G/Z$

Corollary: $G \to S$ reductive gp. & T a max S-torus in G.

① Z_G is the kernel of the adjoint ac: $T \to GL(\mathfrak{g})$

② If $S = \mathrm{Spec}(k)$ for $k = \bar{k}$ then

$$Z_G = \bigcap_{\substack{T' \subset G \\ \max}} T' \quad \text{(scheme theoretic)}$$