Lecture 15.

Reminder: Let $\begin{array}{c} G \\ \downarrow \\ S \end{array}$ be a reductive group scheme

- it admits a maximal torus T étale locally (see lecture 14 / Conrad corollary 3.2.7)

⚠ Actually, using SGA 3, XIV, 3.20 or Conrad exercise 7.3.7. i), we can even make T Zariski locally.

- Assume that T is split

namely, $\exists$ a finite free $\mathbb{Z}$-module M s.t

$$T \simeq D_S(M) := \underset{S\text{-gp}}{\mathrm{Hom}}(M_S, \mathbb{G}_m)$$

Remember that any torus becomes split étale-locally (see Conrad, Proposition B.3.4)

- Splitness def a map of groups $M \to \underbrace{\underset{S\text{-gp}}{\mathrm{Hom}}(T, \mathbb{G}_m)}_{=: X(T)}$

- T acts on $\mathfrak{g} := \mathrm{Lie}(G)$ providing an $\mathcal{O}_S$-linear M-grading $\mathfrak{g} = \bigoplus_{m \in M} \mathfrak{g}_m$

where $t \in T$ acts on $\mathfrak{g}_m$ via multiplication by $m(t)$ (to get convinced work with the corresponding Hopf algebras. See Conrad-Gabber-Prasad A 8.8 for a complete proof)

- Under our assump: (G smooth) one has

$\mathrm{Lie}(G) = W_{\mathfrak{g}}$ and $\mathrm{Lie}(G_R) = \mathrm{Lie}(G)_R$

$\quad \mathrm{Lie}(G_R)(R) \quad \mathrm{Lie}(G)(\mathcal{O}_S) \otimes_{\mathcal{O}_S} R$

$\forall \mathcal{O}_S$-alg R and

the forma° of the weight spaces commutes with base change on S.

this is because the $\mathfrak{g}_m$'s are direct summands of $\mathfrak{g}$ (hence locally free thus flat) (namely vector bundle)

- Nakayama's lemma = If N is a f.g module over a local ring R with max ideal m, any basis of the R/m-vector space N/mN lifts to a minimal set of generators of N.

Conversely every minimal set of generators of M is obtained in this way.

Geometrically, this means here that the weight space decomposi° on the s fibre encodes the weight space dec. on the nearby fibres.

$\Longrightarrow$ By shrinking around s we can arrange that all weight spaces have constant rank. In part the only

characters $m \in M$ for which the weight space (3)
$\mathfrak{g}_m$ is not zero are $m=0$ and elements of
$\Phi = \Phi(G_{\bar{s}}, T_{\bar{s}})$ for the connected reductive geom
fibres $G^\circ_{\bar{s}} \neq 1$.

Remarks: More precisely

① $\mathfrak{g}_0 = \mathrm{Lie}(T)$ as $\mathrm{Lie}(T) =: \mathfrak{t} \subset \mathfrak{g}_0$ as subbundles of $\mathfrak{g}$ with equality on geom fibres over S;

② when $\mathfrak{g}_m \neq 0$ and $m \neq 0$, it is of rank 1

Today: Develop a general theory of root spaces and root groups

From now on: Let S be a non-empty scheme.
Assume that G admits a split maximal torus T over S and fix an isomorphism $T \simeq D_S(M)$ where M is a finite free $\mathbb{Z}$-module.

def: A root for (G,T) is a non zero element $a \in M$ s.t $\mathfrak{g}_a$ is a line bundle. We call such a $\mathfrak{g}_a$ a root space for (G, T, M)

keep in mind that T is assumed to be split!

Remarks: ① Roots can be seen as fibrewise non (4)
trivial characters $T \to \mathbb{G}_m$ corresponding to constant sections of the étale sheaf $\mathrm{Hom}_{S\text{-gp}}(T, \mathbb{G}_m)$ that are fibrewise $\neq 0$ and induce the roots in the classical sense on the geom. fibres.

② While the root spaces $\mathfrak{g}_a$ are line bundles they need not be trivial as such: for instance assume that $S = \mathbb{P}^1_k$ for a field k, let $\mathcal{L} := \mathcal{O}(1)$ over $\mathbb{P}^1_k$ and $G = GL(\mathcal{E})$ where $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}$

This is a Zariski-twisted form of $GL(\mathcal{E})$, the torsor being given by the frame bundle of $\mathcal{E}$
$P = \mathrm{Isom}_{\mathcal{O}_S}(\mathcal{O}_S^{\oplus 2}, E)$ (check it!)

$T = \mathbb{G}_m^2 = D_S(\mathbb{Z}^2) \subset G$ is split, maximal and acts as scaling on both $\mathcal{O}$ and $\mathcal{L}$.
Here $\mathfrak{g} = \mathrm{End}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^* = \underbrace{\mathcal{O}^{\oplus 2}}_{= \mathfrak{t}} \oplus \underbrace{\mathcal{L}^{-1}}_{\mathfrak{g}_a} \oplus \underbrace{\mathcal{L}}_{\mathfrak{g}_{-a}}$

where $a: T \longrightarrow \mathbb{G}_m$
$(c_1, c_2) \longmapsto c_1 / c_2$

non-trivial line bundles as $\mathcal{L}$ is non trivial.

Note that we keep using additive notations for characters of T.

Now that we have built root spaces $\mathfrak{g}_a$ the next step is to build root groups.

Remember the classical case: over an alg closed field k, if $a \in \Phi(G,T)$ and $T_a = (\ker a)^{0}_{red}$ is the unique codim 1 torus in T killed by a then $Z_G(T_a)$ is a connected reductive subgroup of G with Lie alg $\mathfrak{g}^{T_a}$ containing $\mathfrak{g}_a$. Thus the codim 1 torus T_a in T is the max central torus in $Z_G(T_a)$ $\Rightarrow$ $\mathrm{Lie}(Z_G(T_a)) = \mathfrak{g}^{T_a}$ has, as T-roots, $\mathbb{Q}$-multiples of a in $\Phi(G,T) \subset X(T)_{\mathbb{Q}}$ since a is a non trivial character of $T/T_a \simeq G_m$.

In particular $\mathcal{D}(Z_G(T_a))$ is a semi-simple gp having as max torus $T' = (T \cap \mathcal{D}(Z_G(T_a)))^{0}_{red}$.

By the semi-simple rank 1 characteriza- $\mathcal{D}(Z_G(T_a))$ is isomorphic to either SL_2 or PGL_2 and the isomorphism can be chosen to carry T' over the diag torus. In part: the roots for (G,T) that are $\mathbb{Q}$-multiples of a are pairly $\pm a$ and there is a (uniq up to conj) central isogeny $q_a : SL_2 \to \mathcal{D}(Z_G(T_a))$ carrying D onto T' and $\mathcal{U}^+$ onto the root gp $\mathcal{U}_a \subset \mathcal{D}(Z_G(T_a))$ that is k-iso to G_a, normalized by T with $\mathrm{Lie}(\mathcal{U}_a) = \mathfrak{g}_a$.

upper triang matrices

Back to the general case:

Let $a: T \longrightarrow \mathbb{G}_m$ be a root. Since a is fibrewise non trivial its kernel $\ker a$ is S-flat by the fibral flatness criterion and is therefore an S-gp of mult. type.

Conrad B.3.3
Let S be a scheme and H' an S-gp of mult type.
Any fppf closed subgp $H \subset H'$ is of mult. type

Given $f: X \longrightarrow Y$, $g: Y \to S$.
If g is flat, f is finite type and the restriction $f_s: X_s \longrightarrow Y_s$ is flat $\forall s \in S$ then f is flat

and

Lemma: A fibrewise non trivial $a: T \longrightarrow \mathbb{G}_m$ is a root of (G, T) iff so is $-a$, in which case the common kernel $\ker(a) = \ker(-a)$ contains a unique subtorus $T_a = T_{-a}$ of rel. codim 1 in T. The reductive centralizer of the latter $Z_G(T_a)$ has geom fibres with semi-simple rank 1 and $\mathfrak{g}^{T_a} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$

? To def a root gp we need to find a way to "attach" to a a subgroup of (theoretically $\mathcal{D}(Z_G(T_a))$ but at the moment I haven't introduced derived gps in such generalities) $Z_G(T_a)$, which is iso to $\mathbb{G}_a$, normalized by T and whose Lie alg is $\mathfrak{g}_a$.

Remember that: if $S = \mathrm{Spec}(R)$ & N is a proj f.g R-module W_N is represented by $\mathrm{Sym}(N^\vee)$
$\mathrm{Lie}(W_N) \simeq N$ with trivial Lie bracket.

$\leadsto$ if we can induce a S-gp hom
$$W(\mathfrak{g}_a) \longrightarrow G$$
we have a good candidate!

Thm: (Conrad 4.1.4).

Remember $G \supset T \simeq D_S(M)$, $a \in M$ a root, $G \to S$.

Let T act on $W(\mathfrak{g}_a)$ via $t.v = a(t)v$ using the $\mathcal{O}$ structure on $\mathfrak{g}_a$.
There is a unique S-gp homomorphism
$$\exp_a: W(\mathfrak{g}_a) \longrightarrow G$$
inducing the canonical inclusion $\mathfrak{g}_a \hookrightarrow \mathfrak{g}$ on Lie alg & intertwining the T-action on G via conjugation and

the T-action on $W(\mathfrak{g}_a)$ via scaling.

The map $\exp_a$ is also a closed immersion factoring through $Z_G(T_a)$, its formation commutes with base change on S and the multiplica^- map

$$W(\mathfrak{g}_{-a}) \times T \times W(\mathfrak{g}_a) \longrightarrow Z_G(T_a)$$
$$(x', t, x) \longmapsto \exp_{-a}(x')\, t \exp_a(x)$$

is an isomorphism onto an open subscheme $\Omega_a \subset Z_G(T_a)$.

Moreover the semi direct pduct $T \ltimes W(\mathfrak{g}_{\pm a})$ is a a closed S-subgp of G.

Def: The closed subgroup $\exp_a(W(\mathfrak{g}_a)) \subset G$ is called the a-root group for (G, T, M).

Example: $G = Sl_2$ over a base scheme $Spec(R)$

Remember that a 1-param subgp of G is an R-hom $\lambda : \mathbb{G}_m \longrightarrow G$. Let $\lambda(t) = diag(t, 1/t)$ be the standard 1-param subgroup. Remember that $\forall R' \in R$-alg, λ def a unipotent gp $U_G(\lambda)$,

$U_G(\lambda)(R') = \{ g \in G(R') \mid \lim_{t \to 0} \lambda(t) g$ exists & equals $1 \}$

the orbit map $\mathbb{G}_m \longrightarrow G$ def $\forall R' \in R$-Alg & $g \in G(R')$ by

$$\mathbb{G}_m(R') \longrightarrow G(R')$$
$$t \longmapsto t \cdot g$$

extends to $\mathbb{A}^1_S \longrightarrow G$
$\quad$ " $Spec(R)$

Here $U_{Sl_2}(\lambda) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ & $U_{Sl_2}(-\lambda) = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$

corresponding to $\pm a : T \longrightarrow \mathbb{G}_m$ s.t $\pm a(\lambda(t)) = t^{\pm 2}$.

The proof of the thm shows that

$$\exp_a(W_{\mathfrak{g}_a}) = U_{Sl_2}(\lambda) \quad \text{and} \quad \exp_{-a}(W_{\mathfrak{g}_{-a}}) = U_{Sl_2}(-\lambda)$$

Under the identifica^- $W_{\mathfrak{g}_{\pm a}} \cong \mathbb{G}_a$

$\exp_a(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $\exp_{-a}(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$

which "motivates" the exp nota^-

$u_+(z) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$ & $u_-(z) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$

are s t $u_\pm(z)^2 = 0$ & $\exp_{\pm a}(z)$ "=" $1 + n_\pm(z)$.

(the standard trivializa^- are given by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$)

Idea of proof:

① T-equiv + $T_a \curvearrowright W(\mathfrak{g}_a)$ trivially

$\Rightarrow$ if $\exp_a$ is to exist it must factor through $Z_G(T_a)$ $\rightsquigarrow$ Reduce to G with geom fibres of semi-simple rank 1 with T_a central in G and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$

② Uniqueness (once showed) $\Rightarrow$ compatibility with base change = allow to work étale locally

Choose $\lambda : \mathbb{G}_m \longrightarrow T$ corresponding to an elem^t in $X^\vee$ s.t $\langle a, \lambda \rangle \in \mathbb{Z}_{>0}$ (or equiv s.t $a \circ \lambda \in \mathrm{End}_{S\text{-gp}}(\mathbb{G}_m)$ is a constant sec^n)

It may be impossible to arrange that this pairing is 1.

③ Show that $T = Z_G(\lambda)$ by showing equality on the geom fibres

④ Using the aforementioned ac^n $T \curvearrowright W(\mathfrak{g}_a)$ & composing with λ get a $\mathbb{G}_m$-ac^n on $W(\mathfrak{g}_a)$

a pt c of $\mathbb{G}_m$ acts on $W(\mathfrak{g}_a)$ via $a(\lambda(c)) = c^{\langle a, \lambda \rangle}$.

but $\langle a, \lambda \rangle > 0$ so for $H = W(\mathfrak{g}_a)$,

$U_H(\lambda)(R') = \{ a \in H(R') \mid \lim_{t \to 0} t^{\langle a, \lambda \rangle} = 1 \} = H(R')$

$\forall R' \in R\text{-Alg}$

⑤ $\exp_a$ (if it exists) is T-equiv $\Rightarrow$ it must be $\mathbb{G}_m$-equiv $\Rightarrow$ $\exp_a$ must carry H into $\mathcal{U}_G(\lambda)$

But $\mathrm{Lie}(\mathcal{U}_G(\lambda))$ = positive weight space for $\mathbb{G}_m \curvearrowright \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ $\Rightarrow$ $\mathrm{Lie}(\mathcal{U}_G(\lambda)) = \mathfrak{g}_a$.

$\Rightarrow$ If $\exp_a$ is to exist it must factor through an S-hom $W(\mathfrak{g}_a) \longrightarrow \mathcal{U}_G(\lambda)$ that induces an iso on Lie alg.

\+ S-smooth with connected fibres

$\Rightarrow$ $\exp_a$ is an étale hom onto $\mathcal{U}_G(\lambda)$

⑥ Check that if $\exp_a$ is to exist it is actually an iso (check this on the geom fibres)

Summary: If $\exp_a$ is to exist it must be a $\mathbb{G}_m$-equiv iso $W(\mathfrak{g}_a) \simeq \mathcal{U}_G(\lambda)$

⑦ Conversely any such $\mathbb{G}_m$-equiv iso is T-equiv as a map to G as T_a acts trivially on both $W(\mathfrak{g}_a)$ & $\mathcal{U}_G(\lambda)$ and $\mathbb{G}_m \times T_a \longrightarrow T$ is an isogeny of tori.

$(c, t) \longmapsto \lambda(c)t$

$\Rightarrow$ To prove existence & uniqueness of $\exp_a$ it is necessary & sufficient to prove existence & uniqueness of a $\mathbb{G}_m$-equiv iso of S-gps $W(\mathfrak{g}_a) \simeq \mathcal{U}_G(\lambda)$ that induces the identity map on Lie alg.

Once we have $\exp_{\pm a}$ the remaining part of the Th. is immediate as $T = Z_G(\lambda)$ & $\exp_{\pm a}$ carries $W(\mathfrak{g}_{\pm a})$ over $\mathcal{U}_G(\pm\lambda)$. Moreover this identifies $T \ltimes W(\mathfrak{g}_{\pm a})$ with $\underbrace{Z_G(\pm\lambda) \ltimes \mathcal{U}_G(\pm\lambda)}_{= P_G(\pm\lambda)}$

showing that $T \ltimes W(\mathfrak{g}_{\pm a})$ is indeed a closed S-subgp of G.

⑧ Uniqueness of $\exp_a$?

comes from the fact that $W(\mathfrak{g}_a)$ has no non trivial $\mathbb{G}_m$-equiv aut that induces the identity on the Lie alg

(work Zariski locally and have a look at the aut. of $\mathbb{G}_a$)

Indeed an end of $\mathbb{G}_a$ over a ring R is an additive polynomial but $\mathbb{G}_m$-equiv then implies that the polynomial is $x \mapsto cx$, $c \in R$, so the resulting opera- on Lie is scaling by c.