Algebraic Groups. Lecture 8

Reminder: For $G = GL_n$, $G(k)$ is covered by the subgroups $B(k)$ as B varies through the Borel subgroups (that are all $G(k)$-conjugate) and the subset of semi-simple elements in $G(k)$ is covered by the $T(k)$'s as T varies through the max tori (that are all $G(k)$-conj)

Last time we saw that this still hold to be true for every connected linear alg gp G

- In part a connected linear alg group G over $k = \bar{k}$ is a torus when all k-points are semi-simple. Conversely if G contains no non trivial tori then it is unip

$\implies$ Any connected linear algebraic group either admits a strictly upper triangular faithful representa° or it contains a non trivial k-torus.

Today: Attach to a connected reductive k-group G and a maximal torus $T \subset G$ a combinatorial data that determines (G, T) uniquely up to iso.

the so called root datum.

A combinatorial approach: roots & coroots

Fix a k-torus T. Note that $\mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$ via $t \mapsto t^n$

Denote $X(T) = \mathrm{Hom}_{k\text{-gps}}(T, \mathbb{G}_m)$ – character lattice

contravariant in T

finite free $\mathbb{Z}$-modules of rank $\dim T$ (as $T = \mathrm{diag}(\mathbb{G}_m)^r$)

$X_*(T) = \mathrm{Hom}_{k\text{-gps}}(\mathbb{G}_m, T)$ – cocharacter lattice

covariant in T

⚠ elem^T of char. & cochar are denoted additively e.g. $T \underset{b}{\overset{a}{\rightrightarrows}} \mathbb{G}_m$, $a+b : t \mapsto a(t)b(t)$, $-a : t \mapsto 1/a(t)$

There is a perfect pairing,

$$X(T) \times X_*(T) \longrightarrow \mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$$

$$a, \lambda \longmapsto a \circ \lambda$$

$$\begin{array}{cc} T & \mathbb{G}_m \\ \downarrow & \downarrow \\ \mathbb{G}_m & T \end{array}$$

(denote $\varphi : X_*(T) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$, $\lambda \longmapsto (a \mapsto a \circ \lambda)$) the corresponding iso of $\mathbb{Z}$-mod.

Note that

$$X_*(T) \otimes_{\mathbb{Z}} k^\times \simeq T(k)$$
$$\lambda \otimes c \longmapsto \lambda(c)$$

This really comes from the existence of the perfect pairing as it implies $X_*(T) \otimes_{\mathbb{Z}} k^\times = \mathrm{Hom}(X(T), k^\times)$.

As a k-scheme T can be reconstructed from its character lattice via

$$X_*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m \longrightarrow T$$

$\forall A \in k\text{Alg}$

$$(X_*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m)(A) = X_*(T) \otimes_{\mathbb{Z}} A^\times = \mathrm{Hom}_{\mathbb{Z}}(X(T), A^\times)$$

In other words the functors

$$\left\{ \begin{array}{c} \text{Finite free} \\ \mathbb{Z}\text{-mod} \end{array} \right\} \longrightarrow \{ \text{tori} \}$$
$$M \longmapsto M^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m$$
$$X_*(T) \longleftarrow T$$

are inverse anti-equiv.

Prop: Let T be a torus and V be a finite dim linear rep of T over k. $\forall a \in X(T)$ denote

$$V_a := \{ v \in V \mid t \cdot v = a(t) v \ \forall t \in T(k) \}$$

The a-weight space

Then

$$\bigoplus_{a \in X(T)} V_a \longrightarrow V \text{ is an iso}$$

In other words, there is a bij corresp. between

$$\left\{ \begin{array}{c} X(T)\text{-graded} \\ k\text{-vector spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{linear rep} \\ \text{of } T \end{array} \right\}$$

def: If $V_a \neq 0$ then a is a **weight** of the representation.

Lemma: $\forall\, a \in X(T)$ one has $(V^*)_a = (V_{-a})^*$ where V^* is the dual rep space. In part the set of weights of a self dual rep of T is stable under nega°.

E.g: $T := \mathbb{G}_m$, $X(T) = \mathbb{Z}$

and linear rep of $\mathbb{G}_m$ $\xleftrightarrow{1:1}$ $\mathbb{Z}$-graded vector spaces

a k-linear hom $\mathbb{G}_m \longrightarrow GL(V)$

$$V = \bigoplus_{n \geq 0} V(n)$$

as $t \in \mathbb{G}_m$ acts on $V(n)$ via t^n-scaling

and analogously for $T = \mathbb{G}_m^r$, $\mathbb{G}_m^r$-actions can be described using $\mathbb{Z}^r$-gradings

A first step towards the generaliza° to the relative settings ...

Let's continue with the example of a linear $\mathbb{G}_m$-action: $\mathbb{G}_m \longrightarrow GL(V)$, choose $v \in V$ and use the above equiv to decompose $v = \sum_{n \in \mathbb{Z}} v_n$ with $v_n \in V(n)$ so that $t.v = \sum_{n \in \mathbb{Z}} t^n v_n$

Consider the orbit map

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & V \\ \mathrm{Spec}(k[t,t^{-1}]) \ni t & \longmapsto & t.v \end{array}$$

seen as an affine space over k

The latter factors through $\mathbb{A}^1 = \mathrm{Spec}(k[t])$ iff $v \in V_{\geq 0} := \bigoplus_{n \geq 0} V(n)$ in such cases the image of $0 \in \mathbb{A}^1(k) = k$ under the extension $\mathbb{A}^1 \to V$ is denoted $\lim_{t \to 0} t.v = v_0$ (so the space of $v \in V$ for which $\lim_{t \to 0} t.v$ exists & vanishes is $V_{>0} := \bigoplus_{n > 0} V(n)$.

More generally if $\mathbb{G}_m \times G \longrightarrow G$ is an action of $\mathbb{G}_m$ on a (separated) group scheme G over a base scheme S, for any $g \in G(S)$ we say that $\lim_{t \to 0} t.g$ exists if the orbit map extends through

namely when the identity section is a closed immersion

this is automatic when we work over a field.

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & G \\ \mathbb{A}^1_S \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad t \longmapsto t.g$$

Note that if such an extension exists it is unique as G is separated and $k[t] \subset k[t, t^{-1}]$ $\forall k \in \mathrm{Ring}$ (write the corresponding Hopf alg diag)

The image of 0 in $G(S)$ under the extension morphism $\mathbb{A}^1_S \longrightarrow G$ is denoted $\lim_{t \to 0} t.g$

Example $G = GL(V)$ $\quad T = D_n = \{\text{diagonal matrices}\}$
$= \mathbb{G}_m^n$.

so $t \in T$ is of shape $\mathrm{diag}(c_1, \dots, c_n)$. Note that $T(k) = Z_{G(k)}(T)$, so that T is maximal. Moreover show (exercise!) that $X(T) := \mathrm{Hom}_{k\text{-gp}}(T, \mathbb{G}_m)$

$$= \bigoplus \mathbb{Z} e_i$$

where $e_i : T \longrightarrow \mathbb{G}_m$
$\mathrm{diag}(c_1, \dots, c_n) \longmapsto c_i$

and $X_*(T) := \mathrm{Hom}_{k\text{-gp}}(\mathbb{G}_m, T)$

$$= \bigoplus \mathbb{Z} e_i^\vee$$

where $e_i^\vee : \mathbb{G}_m \longrightarrow T$
$c \longmapsto \mathrm{diag}(c_1, \dots, c_n)$

with $c_j = \begin{cases} 1 & \text{if } j \neq i \\ c & \text{if } j = i \end{cases}$

- The action of T on G by conjugation induces, for any $\lambda \in X_*(T)$, an action of $\mathbb{G}_m$
- This provides a canonical decomposi²
$$\mathfrak{g} = \bigoplus V_{a_i}$$

where $\{a_1, \dots, a_n\}$ is the FINITE set of pairwise $\neq$ weights for the $\mathbb{G}_m$-action on V via λ (note $m \geq 0$). By the preceding ex $a_i(t) = t^{\alpha_i}$ for a unique $\alpha_i \in \mathbb{Z}$. wlog the a_i's are such that the α_i's are strictly decreasing in i. This defines a filtra° by subspaces $F_j = \bigoplus_{i \leq j} V_{a_i}$ and one can choose. Choose an ordered k-basis $\{v_1, \dots, v_n\}$ of V adapted to this filtration: each v_r is in a weight space $V_{\chi_{i_r}}$ and $i_1 \leq \dots \leq i_n$.

For such a basis one has $G = GL_n$ and $\lambda(t) = \mathrm{diag}(t^{e_1}, \dots, t^{e_n})$ for $e_r = \alpha_{i_r}$ and $e_1 > \dots > e_r$.

Now if $g = (m_{rs}) \in GL(V) \simeq GL_n$ then

$$\lim_{t \to 0} \lambda(t) . g \lambda(t)^{-1} = (t^{e_r - e_s} m_{rs}) = (t^{\alpha_{i_r} - \alpha_{i_s}} m_{rs})$$

exists iff $m_{rs} = 0$ whenever $i_r > i_s$

and $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} = 1$ iff $m_{rs} = 0$ whenever $i_r > i_s$ and $m_{rs} = 1$ whenever $\alpha_{i_r} = \alpha_{i_s}$

In particular, when the e_i's are all $\neq$ the upper triangular subgp $B_n \subseteq G$ represents the functor of points g of G for which $\lim_{t \to 0} \lambda(t). g$ exists and U_n represents the functor of points for which $\lim_{t \to 0} \lambda(t). g$ exists and equals 1.

This gives a mechanism for constructing B & U entirely in terms of G and the $\mathbb{G}_m$-ac° on it with no reference to no° such as solvability or unip. that are not convenient to work with in a relative settings.

For arbitrary $\lambda \in X_*(T)$ note that the functor of points for which $\lim_{t \to 0} \lambda(t). g$ exists is represented by a parab subgp $P_G(\lambda)$ and its unip. radical represents the functor of pts for which $\lim_{t \to 0} \lambda(t). g$ exists and equals 1.

Back to the filtra° by the F_j's, note that

$$P_G(\lambda) = \{ g \in G \mid \lim_{t \to 0} \lambda(t) . g \text{ exists} \}$$

indeed corresp^ds to aut of V respecting the increasing filtra° by the F_j's.

Roots and coroots.

Let G be a connected reductive k-gp
$\cup$
T a max torus

⚠ For a smooth k-gp H we denote $\mathfrak{h} := \mathrm{Lie}(H) = \underline{\mathrm{Lie}}(H)(k)$
its Lie alg which is Zariski dense in $\underline{\mathrm{Lie}}(H) \cong W_{\mathrm{Lie}(H)}$

Note that the adjoint T ac° on $\mathfrak{g}$ is completely reducible (as for any linear rep of a torus)

⚠ In char $p>0$ the whole ac° of G may fail to be semi simple

E.g if $\mathrm{char}(k)=p$ and $G = SL_p$ then

$\mathfrak{g} = \mathfrak{sl}_p =$ matrices of trace zero $\supset \underbrace{\{\lambda \mathrm{Id} \mid \lambda \in k\}}_{\text{stabilized by } G}$

but has no G-equiv complement.

Goal: Study $\mathfrak{g}$ enriched with the T-module structure. It decomposes into weight spaces as:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{a \in \Phi} \mathfrak{g}_a \right)$$

exercise sheet 2

$\mathrm{Lie}(Z_G(T)) = \mathfrak{g}^T = \mathfrak{g}_0$

$\Phi := \Phi(G,T) \subset X(T) \setminus \{0\}$ the set of weights for the T-action

Note that Φ is FINITE and

$$\mathfrak{g}_a := \{ x \in \mathfrak{g} \mid t.x = a(t)x \ \forall t \in T(k) \} \neq 0$$

is
- 1-dimensional (Borel 13.18 4(b))
- stable under neg° (Borel 13.18.1 and 13.18.4(a))

⚠ This doesn't mean that $\mathfrak{g}$ is self dual as a G-representation see Conrad exercise 1.6.4 for a counter-ex

Note: $\mathfrak{g}_\alpha$ is 1-dim $\forall \alpha \in \Phi(G,T) \Rightarrow$ the characteristic polynomial of the T-ac° on $\mathfrak{g}$ is given by

$$(x-1)^r \prod_{a \in \Phi} (x - a(t)) \quad \text{for } t \in T(k)$$

r = the rank of G = dim of T

This explains the terminology: elements of Φ are called the roots of (G,T) and the $\mathfrak{g}_a$'s are the root spaces in $\mathfrak{g}$.

Ex: Back to GL_n. Then $\mathfrak{g} = M_n(k)$, T = diag matrices

The roots are the characters

$$a_{ij} : T \longrightarrow \mathbb{G}_m, \quad \mathrm{diag}(c_\ell) \longmapsto c_i / c_j \quad \text{for } 1 \leq i \neq j \leq n$$

The corresp. root spaces are the

$\mathfrak{g}_{a_{ij}} \subset \mathfrak{N}_n(k)$

$\quad\|$

(m_{rs}) with $m_{rs} = 0$ if $(r,s) \neq (i,j)$

Example: $G = SL_2$ $\quad T = \{\operatorname{diag}(c, 1/c)\}$ is max $=: D$

$\mathfrak{g} = \mathfrak{sl}_2$

is identified with $\mathbb{G}_m$ via

$\lambda : \mathbb{G}_m \longrightarrow T$

$c \longmapsto \begin{pmatrix} c & \\ & 1/c \end{pmatrix}$

a basis is given by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

kH coincides with $\mathfrak{t}$ while kE and kF are the root spaces (resp $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$)

Def: (Root group) For $a \in \Phi$, $\exists!\ \mathcal{U}_a \subset G$ subgp normalized by T s.t. $\mathcal{U}_a \cong \mathbb{G}_a$ and $\operatorname{Lie}(\mathcal{U}_a) = \mathfrak{g}_a$ (Borel. 13.18 (4d))

The latter is the root gp associated to a

? Explicit action of T on $\mathcal{U}_a$?

Exercise: Show that all automorphisms of $\mathbb{G}_a$ over $k = \bar{k}$ is nothing but a $k^\times$-scaling.
Then $T \curvearrowright \mathcal{U}_a \cong \mathbb{G}_a$ via $t.x = \alpha x$, $\alpha \in k^\times$ such that the derived ac is given by $a(t)$ this forces $\alpha = a(t)$.

Example: $G = SL_2$, $T = D$

$\mathcal{U}^{\pm} := \mathcal{U}_{\pm 2}$ = strictly upper (resp lower) triang unip subgp.

(same for $G = PGL_2$, $T = \bar{D}$ and $\bar{\mathcal{U}}^{\pm}$

That SL_2 is generated by $\mathcal{U}^{\pm}$ admits the following generaliza:

Lemma: Let $T_a := (\ker a)^{\circ}_{\text{red}}$ be the unique codim 1 torus in T killed by $a \in \Phi$.
The root gps $\mathcal{U}_{\pm a}$ generate $\mathcal{D}(Z_G(T_a))$ which is a closed subgp admitting PGL_2 as an isogenous quotient

Proof: Lemma 1.2.6 Conrad's notes.

? How do we define coroots ?

A starting point: The SL_2 example.

$T = D$ identifies with $\mathbb{G}_m$ via

$$\mathbb{G}_m \longrightarrow T \qquad c \longmapsto \begin{pmatrix} c & \\ & 1/c \end{pmatrix}$$

roots for the D-ac° are

$$c^2 : T \longrightarrow \mathbb{G}_m \qquad \begin{pmatrix} c & \\ & 1/c \end{pmatrix} \longmapsto c^2$$

$$c^{-2} : T \longrightarrow \mathbb{G}_m \qquad \begin{pmatrix} c & \\ & 1/c \end{pmatrix} \longmapsto c^{-2}$$

This admits the following generaliza°:

Thm: For each $a \in \Phi(G, T)$ $\exists$ a hom.
$\varphi_a : SL_2 \to G$ carrying D into T and $U^{\pm}$ isomorphically onto the root gps $U_{\pm a}$

Such a φ_a is an isogeny onto $\mathcal{D}(Z_G(T_a))$ with $\ker \varphi_a \subset \mu_2$, it is uniq up to $T(k)$-conjuga° on G.

Proof: Conrad Thm 1.2.7.

$$\begin{array}{ccc} SL_2 & \overset{\varphi_a}{\dashrightarrow} & G \\ \downarrow q & \swarrow f\,? & \\ PGL_2 & & \end{array}$$

canonical isogeny

Idea: Replace (G, T) by $(\mathcal{D}(Z_G(T_a)), T'_a)$

semi-simple of rk 1

an isogeny complement to T_a given by $T \cap \mathcal{D}(Z_G(T_a)^0)_{red}$

hence G is iso to either SL_2 or PGL_2

f is def as $f : G \longrightarrow \mathrm{Aut}_{\mathbb{P}^1_k/k} = PGL_2$ associated to the left transla° action of G on G/B

Def: The coroot associated to (G, T, a) is the cocharacter $a^\vee : \mathbb{G}_m \longrightarrow D \longrightarrow T$

$$c \longmapsto \varphi_a\begin{pmatrix} c & \\ & 1/c \end{pmatrix}$$

by what preceeds it is unaffected by $T(k)$-conj on G, namely it is intrinsic.

Note:

$a^\vee$ is a param. (with kernel 1 or μ_2) of the 1-dim torus $(T \cap \mathcal{D}(Z_G(T_a)))^\circ_{red}$ that is an isogeny complement to T_a in T.

Ex ① $G = SL_2$ $T = D$

$\varphi_a = id$ and $a^\vee(c) = \mathrm{diag}\begin{pmatrix} c & \\ & 1/c \end{pmatrix}$

$\langle a, a^\vee \rangle = 2 \in \mathrm{End}(G_m) = \mathbb{Z}$

namely $a(a^\vee(c)) = c^2$

as $\begin{pmatrix} 1 & \\ & 1/c \end{pmatrix} \underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}_{U_a} \begin{pmatrix} 1 & \\ & 1/c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c^2 x \\ 0 & 1 \end{pmatrix}$

& the adj ac^n of $a^\vee(c)$ on $\mathfrak{g}_a = \mathrm{Lie}(U_a)$ is scaling by c^2.

② $G = PGL_2$ $T = \bar{D}$ $G_m \simeq T$ via

$c \mapsto \begin{pmatrix} c & \\ & 1 \end{pmatrix} \bmod G_m$

$\bar{a} : T \longrightarrow G_m$ is a root whose root space consists of upper triang nilp matrices in $\mathfrak{pgl}_2$ (same for $-\bar{a}$ with lower).

So $\varphi_{\bar{a}} : SL_2 \longrightarrow G$ is the canonical proj

15. Def: A root datum is a 4-tuple $(X, \Phi, X^\vee, \Phi^\vee)$