ALGEBRAIC GROUPS

PROBLEM SHEET 1

Please hand in your solutions to the following problems by 2026-02-20, either by email or in my mailing box at the department or during the lectures. You are encouraged to discuss the problems with your colleagues, but must hand in your own solutions. You are welcome to ask questions.

Unless otherwise specified, k is an arbitrary unital commutative ring. When dealing with Hopf algebras, $\mathfrak{m}$, $\mathfrak{e}$ and $\mathfrak{i}$ always denote the comultiplication, counit and antipode, respectively.

Problem 1. Let $X = \mathrm{Spec}(A)$ be an affine scheme, where A is a ring. Show that for any field K there is a bijection between the set of morphisms $\mathrm{Spec}(K) \to X$ and the set of pairs (x, i) where $x \in X$ and $i : \kappa(x) \to K$ is a morphism of fields.

Problem 2. Show the following assertions:

a) the empty k–functor is not representable,
b) the functor $\mathbf{X} : k\text{-}\mathbf{Alg} \to \mathbf{Set}$ defined by $\mathbf{X}(R) = \Sigma$ for all $R \in k\text{-}\mathbf{Alg}$, with Σ a finite set with at least two elements is not representable,
c) the functor $\mathbb{G}_m : \mathbf{Ring} \to \mathbf{Set}$ that sends a ring to its set of units is representable (give a representing object!). (Note that $\mathbb{G}_m$ is actually a group functor).

Problem 3. Let k be a field and let $k' = k[\varepsilon]/(\varepsilon^2)$. Consider the k'-scheme $\mathbf{X}$ whose R-points are given, for any $R \in \mathbf{k'}\text{-}\mathbf{Alg}$, by

$$\mathbf{X}(R) = \{x \in R \mid x^2 = \varepsilon\}.$$

Compute the k-rational points of $\Pi_{k'/k}(\mathbf{X})$.

Problem 4. The Krull dimension of k, denoted $\dim(k)$, is the supremum on the length r (careful: I really do mean r here, not $r+1$) of chains $p_0 \subset p_1 \subset \cdots \subset p_r$ of prime ideals of k. The dimension of an affine scheme $\mathbf{X} = \mathrm{Spec}(A)$, where A is a ring, is the Krull dimension of A.

a) Show that if k is a field then $\dim(k) = 0$, when $k = \mathbb{Z}$ then $\dim(k) = 1$ and when k is any nœtherian ring (that is, any ideal is finitely generated), if t is an indeterminate then $\dim(k[t]) \geq \dim(k) + 1$ (actually when k is nœtherian this is an equality).
   <u>Hint:</u> To show the last statement you may use that if k is nœtherian, so is $k[t]$.
b) Deduce from a) that the dimension of the product is not the sum of the dimension of each factor. However, note that this turns out to be true if we replace dim by $\dim^*(X) = \dim(X) - \dim(\mathrm{Spec}(\mathbb{Z}))$. This is another justification for the construction of a fibre product in the category of affine schemes.

Problem 5. Show that an affine group scheme $\mathbf{G}$ is commutative (i.e. $\mathbf{G}(R)$ is abelian for all $R \in k\text{-}\mathbf{Alg}$) if and only if the Hopf algebra $A = k[\mathbf{G}]$ is cocommutative (i.e. $\tau\mathfrak{m} = \mathfrak{m}$, where τ is the twist $a \otimes b \mapsto b \otimes a$).

Problem 6.

a) Determine the Hopf algebra structure on $k[x]$ corresponding to the additive group $\mathbf{G}_{\mathrm{a}}$, i.e. find $\mathfrak{m}$, $\mathfrak{e}$ and $\mathfrak{i}$.
b) Verify that the Hopf algebra structure on $k[x, x^{-1}]$ corresponding to the multiplicative group $\mathbf{G}_{\mathrm{m}}$ is defined by

$$\mathfrak{m}(x) = x \otimes x, \quad \mathfrak{e}(x) = 1, \quad \mathfrak{i}(x) = x^{-1}.$$

.

c) Show that the ideal $(x^n - 1)$ is a Hopf ideal in $k[x, x^{-1}]$.

Problem 7. Let Γ be a finite abstract group. Recall that we have defined the constant group scheme $\mathbf{Hom}_A$ corresponding to Γ. Show that $\mathbf{Hom}_A(R) \simeq \Gamma$ whenever $R \in k\text{-}\mathbf{Alg}$ is connected (i.e. has no other idempotents (elements $r \in R$ such that $r^2 = r$) than 0 and 1).

Problem 8. Let k be a field and $\mathbf{G}$ be an affine k-group scheme of finite presentation. Let then $\mathbf{H} \leq \mathbf{G}$ be a closed subgroup that acts on $\mathbf{Lie}(\mathbf{G})$ via the adjoint representation. Denote by $\mathbf{Z_G}(\mathbf{H})$ the centraliser of $\mathbf{H}$ in $\mathbf{G}$. Show that $\mathbf{Lie}(\mathbf{Z_G}(\mathbf{H})) = \mathbf{Lie}(\mathbf{G})^{\mathbf{H}}$.