HOMEWORK SET 2

HW1. Let X be a simply-connected non-contractible finite-dimensional CW complex. Show that ΩX has nontrivial homology in infinitely many degrees. (Hint: Use the Hurewicz theorem [to be proved in class Oct 16] to see that there is a prime p such that X has nontrivial mod p homology. Analyze the mod p Serre spectral sequence of the path-loop fibration $\Omega X \to PX \to X$.)

HW2. Consider the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and take the quotient by the antipodal map on the total space. We obtain a fibration sequence

$$S^1 \to \mathbb{RP}^{2n+1} \to \mathbb{CP}^n,$$

since S^1 modulo its antipodal involution is S^1. Study the Serre spectral sequence of this fibration, both for $\mathbb{Z}$-coefficients and $\mathbb{Z}/2$-coefficients. Determine the differentials in the spectral sequence to compute the cohomology groups $H^\bullet(\mathbb{RP}^{2n+1}, \mathbb{Z})$ and $H^\bullet(\mathbb{RP}^{2n+1}, \mathbb{Z}/2)$. (Hint: for both coefficients use that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$ and the universal coefficient theorem to determine the first differential, and then use multiplicativity.)

Does the ring structure you get on the E_∞ page coincide with the ring structure on $H^\bullet(\mathbb{RP}^{2n+1}, \mathbb{Z})$ resp. $H^\bullet(\mathbb{RP}^{2n+1}, \mathbb{Z}/2)$? (Hint: Poincaré duality says that if M is a connected closed manifold of dimension d, orientable or not, then $H^k(M, \mathbb{Z}/2) \otimes H^{d-k}(M, \mathbb{Z}/2) \to H^d(M, \mathbb{Z}/2) \cong \mathbb{Z}/2$ is a perfect pairing.)

Deadline: 2025–10–23. If you have used any resources outside the course literature/lecture notes, please indicate this in your solution. Similarly if you have discussed the problems with another student, or used AI. Hand in your solutions by e-mail to: dan.petersen@math.su.se