# ALGEBRAIC GROUPS

PROBLEM SHEET 2

Please hand in your solutions to the following problems by 2026-03-20, either by email or in my mailing box at the department or during the lectures. You are encouraged to discuss the problems with your colleagues, but must hand in your own solutions. You are welcome to ask questions.

**Problem 1.** Let $k$ be a field and $\mathbf{G}$ be an affine $k$-group scheme of finite presentation. Let then $\mathbf{H} \leq \mathbf{G}$ be a closed subgroup that acts on $\mathcal{L}\mathbf{ie}(\mathbf{G})$ via the adjoint representation. Denote by $\mathbf{Z_G}(\mathbf{H})$ the centraliser of $\mathbf{H}$ in $\mathbf{G}$. Show that $\mathcal{L}\mathbf{ie}(\mathbf{Z_G}(\mathbf{H})) = \mathcal{L}\mathbf{ie}(\mathbf{G})^{\mathbf{H}}$.

**Problem 2.** Let $k$ be a field of characteristic 2 and let $\mathbf{G} = \mathbf{SL}_2$.
Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}\mathbf{ie}(\mathbf{G})(k)$. Show that the centraliser $\mathbf{Z_G}(x)$ is not smooth.

**Problem 3.** Let $\mathbf{S}$ be an affine scheme and $\mathbf{G}$ be a smooth $\mathbf{S}$-group scheme of finite presentation.

(1) Let $\mathfrak{h} \subseteq \mathbf{Lie}(\mathbf{G})$ be a Lie subalgebra functor, show that $\mathbf{N}_{\mathcal{L}\mathbf{ie}(\mathbf{G})}(\mathfrak{h}) = \mathcal{L}\mathbf{ie}(\mathbf{N_G}(\mathfrak{h}))$;
(2) let $\mathbf{H} \subseteq \mathbf{G}$ be a closed subgroup. Show that $\mathbf{N_G}(\mathbf{H}) \subseteq \mathbf{N_G}(\mathcal{L}\mathbf{ie}(\mathbf{H}))$.
(3) In what follows $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field $k$ and consider the $k$-group scheme $\mathbf{GL}_3$. Let $\mathbf{B} \subset \mathbf{GL}_3$ be the Borel subgroup of upper triangular matrices. Let $\mathbf{V}$ be the unipotent radical of $\mathbf{B}$.
Let $\mathbf{U}$ be the smooth subgroup whose $R$-points, for any $k$ algebra $R$, are
$$\mathbf{U}(R) := \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & t^p \\ 0 & 0 & 1 \end{pmatrix} \mid t \in k \right\}.$$
Show that $\mathbf{N_V}(\mathbf{U})$ is not smooth.

**Problem 4.** Let $k$ be a field, show that $\mathfrak{End}(\mathbf{G}_{m,k}) = \mathbb{Z}$. Note that the result is actually true when $k$ is any local ring.

**Problem 5.** The aim of this exercise is to show that the cokernel of a morphism of $k$-group schemes needs not be representable. Consider the Kummer morphism $f_n : \mathbf{G}_{m,k} \to \mathbf{G}_{m,k}$ given, for any $k$-algebra $R$ by $f_{n,R} : \mathbf{G}_m(R) \to \mathbf{G}_m(R),\ x \mapsto x^n$ for $n \geq 2$ and $k = \mathbb{C}$. Show that there does not exist an affine $\mathbb{C}$-group scheme such that there is a four terms exact sequence of $\mathbb{C}$-functors (exactness should be understood in the strong way here, namely for any $k$-algebra $R$...)
$$1 \to \mu_n \to \mathbf{G}_m \xrightarrow{f_n} \mathbf{G}_m \to \mathbf{G} \to 1.$$

**Problem 6.** We saw during the lectures that if $k$ is a an algebraically closed field of characteristic 0 a linear algebraic $k$-group is reductive if and only if any representation of $G$ is semisimple, this may fail in positive characteristic. The aim of this exercise is to exhibit a counter example. Let $k$ be an algebraically closed field of characteristic $p > 0$ and consider the $k$-reductive group $\mathbf{SL}_p$ and its adjoint action on its Lie algebra. Find a non-empty subalgebra of $\mathcal{L}\mathbf{ie}(\mathbf{SL}_p)$ on which this action is trivial.

**Problem 7.** Let $\mathbf{G}$ be a reductive group defined over an algebraically closed field $k$, denote by $\mathfrak{g}$ its Lie algebra and let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. Let $X(\mathbf{T})$ be the characters of $\mathbf{T}$ and $X_*(\mathbf{T})$ the cocharacters of $\mathbf{T}$. Denote by $\Phi$ the set of the roots associated to the action of $\mathbf{T}$ on $\mathfrak{g}$. Let $\lambda \in X_*(\mathbf{T})$ and define the subset $\Phi_{\lambda \geq 0} \subset \Phi$ as

$$\Phi_{\lambda \geq 0} := \{a \in \Phi \mid (a, \lambda) \geq 0\},$$

where $(.,.) : X(\mathbf{T}) \times X_*(\mathbf{T}) \to \mathbb{Z}$ is the perfect pairing defined during the lecture.

(1) Show that $\Phi_{\lambda \geq 0}$ satisfies the properties of what is usually defined as a *parabolic subset of roots*, namely show that:
- for any cocharacter $\lambda \in X_*(\mathbf{T})$, the root system $\Phi$ satisfies
$$\Phi = \Phi_{\lambda \geq 0} \cup -(\Phi_{\lambda \geq 0});$$
- $\Phi_{\lambda \geq 0}$ is a *closed subset* of $\Phi$, namely if $a,\ b \in \Phi_{\lambda \geq 0}$ and $a + b \in \Phi$ then $a + b \in \Phi_{\lambda \geq 0}$.

(2) Let $\mathbf{P_G}(\lambda)$ be the parabolic subgroup defined by $\lambda$: for any $k$-algebra $R$
$$\mathbf{P_G}(\lambda)(R) := \{g \in \mathbf{G}(R) \mid \lim_{t \to 0} \lambda(t) \cdot g \text{ exists.}\}$$
(a) Show that $\mathbf{T}(k) \subset \mathbf{P}_G(\lambda)(k)$. As these groups are smooth and connected (for the parabolic subgroup this comes from the lecture) and $k$ is algebraically closed, this is enough to conclude that $\mathbf{T} = \mathbf{P_G}(\lambda)$.
(b) Show that $\mathbf{Lie}(\mathbf{P_G}(\lambda)) = \{g \in \mathfrak{g} \mid \lim_{t \to 0} \lambda(t) \cdot g \text{ exists}\}$.
(c) Using the weight space decomposition of the action of $\mathbf{T}$ on $\mathfrak{g}$, show that this action reduces to an action of $\mathbf{T}$ on $\mathbf{Lie}(\mathbf{P_G}(\lambda))$. In particular, show that the resulting root space decomposition is given by
$$\mathbf{Lie}(\mathbf{P_G}(\lambda)) := \mathfrak{t} \oplus \Big(\bigoplus_{a \in \Phi_{\lambda \geq 0}} \mathfrak{g}_a\Big).$$