ALGEBRAIC GROUPS

PROBLEM SHEET 3

Please hand in your solutions to the following problems by 2026-04-24, either by email or in my mailing box at the department or during the lectures. You are encouraged to discuss the problems with your colleagues, but must hand in your own solutions. You are welcome to ask questions.

Problem 1. If $\mathcal{T}$ and $\mathcal{T}'$ are two Grothendieck topologies on a category $\mathcal{C}$, we say that $\mathcal{T}$ is *finer* or *stronger* than $\mathcal{T}'$ if every $\mathcal{T}'$-covering is a $\mathcal{T}$-covering. Show that:

(1) on k-**Aff**, fpqc is stronger than fppf, which is stronger than étale, which is stronger than Zariski;
(2) all the previous refinements are strict, in the sense that there is an fpqc covering that is not an fppf covering, etc.

Problem 2. Show that there is indeed a Zariski sheaf that is not an étale sheaf, an étale sheaf that is not an fppf sheaf, and an fppf sheaf that is not an fpqc sheaf.[1]

Hint: This should not be tricky once you have identified what kind of failure you are looking for examples of.

Problem 3. Let $\mathbf{T}$ be a scheme and $\{f_i : \mathbf{T}_i \to \mathbf{T}\}_{i\in I}$ be a family of morphisms of schemes with target $\mathbf{T}$. Show that the following are equivalent:

(1) the family $\{f_i : \mathbf{T}_i \to \mathbf{T}\}_{i\in I}$ is an fpqc covering,
(2) setting $\mathbf{T}' := \amalg_{i\in I}\mathbf{T}_i$ and $f := \amalg_{i\in I} f_i$, the family $\{f : \mathbf{T}' \to \mathbf{T}\}$ is an fpqc covering.

Problem 4. Let k be a commutative unital ring and let $\mathbf{E}$ and $\mathbf{F}$ be two k-functors that are fppf-sheaves. Show that:

(1) $\mathbf{Hom}(\mathbf{E},\mathbf{F})$ and $\mathbf{Isom}(\mathbf{E},\mathbf{F})$ are fppf-sheaves,
(2) $\mathbf{G} := \mathbf{Aut}(\mathbf{E})$ is an fppf-sheaf of groups, that acts on $\mathbf{Isom}(\mathbf{E},\mathbf{F})$ on the right by precomposition, and
(3) if $\mathbf{E}$ is fppf-locally isomorphic to $\mathbf{F}$, then under this action, $\mathbf{Isom}(\mathbf{E},\mathbf{F})$ is a $\mathbf{G}$-torsor.

Problem 5. Show that the sheafification of a sheaf $\mathbf{F}$ is the sheaf $\mathbf{F}$ itself.

Problem 6. Let $\mathbf{G}$ be an affine group scheme defined over a commutative unital ring k acting and let $\mathbf{E}$ be a $\mathbf{G}$-torsor which is also an affine k-scheme (the definition of a $\mathbf{G}$-torsor doesn't require $\mathbf{E}$ to be representable).

(1) Denote by $f : \mathbf{E} \to \mathrm{Spec}(k)$ the structure morphism of $\mathbf{E}$. Show that the torsor $\mathbf{E}$ is trivial if and only if f has a section (which is equivalent to say that X has a k-rational point).
(2) Assume that k is a field which contains an element t that is not a square. Assume moreover that $\mathbf{E} \subset \mathbb{A}_1$ is defined by $k[\mathbf{E}] = k[x]/(x^2 - t)$. Show that $\mathbf{E}$ is a non-trivial $\boldsymbol{\mu}_2$-torsor for the action induced by that on $\mathbb{A}_1$ by multiplication.

Problem 7. Let $\mathbf{X} = \mathrm{Spec}(\mathbb{Q}(i))$ (seen for instance as a $\mathbb{Q}$-scheme). Show that $\mathbf{X}$ is connected but $\mathbf{X}_{\mathbb{C}}$ is not.[2]

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[1] This exercise shows that even if a topology is strictly finer than another the categories of sheaves with respect to the two topologies may still coincide.

[2] Let X be an affine scheme defined over a field k. It is *geometrically connected* if the scheme $X_{k'}$ is connected for any field extension k'/k.

Problem 8. Let k be a field and let $\mathbf{X} = \mathbf{Hom}_A$ be a k-group scheme of finite type (namely A is a k-algebra of finite type). The k-algebra A is isomorphic to a finite product $\prod_i A_i$ of connected k-algebras of finite type. Let $\mathbf{X}_i = k[A_i]$, one can show that the $\mathbf{X}_i$'s are the connected components of the underlying topological space of $\mathbf{X}$. Note that any separable subalgebra of finite rank $K \subset A_i$ is a field, and if $\mathfrak{m}_i$ is a maximal ideal of A_i, any such K satisfies $[K : k] \leq [A_i/\mathfrak{m}_i : k]$. The directed set defined by these K has a maximal element A_{is}. Set $A_s = \prod_i A_{is}$, the inclusion $A_s \subset A$ induces a morphism of schemes $\gamma_{\mathbf{X}} : \mathbf{X} \to k[A_s] =: \boldsymbol{\pi}_0(\mathbf{G})$.

(1) Show that $\boldsymbol{\pi}_0$ is étale, faithfully flat and its fibres are the connected components of $\mathbf{X}$.

(2) What precedes can be extended to any affine k-group scheme $\mathbf{G}$ and it can be shown that there is a unique group scheme structure on $\boldsymbol{\pi}_0(\mathbf{G})$ such that $\gamma_{\mathbf{G}}$ is a morphism of group schemes. The connected component of $\mathbf{G}$ is defined by $\mathbf{G}^0 := \mathbf{ker}(\gamma_{\mathbf{G}})$.[3] What are the connected components of $\boldsymbol{\mu}_3$ seen as a $\mathbb{R}$-group scheme? Show that one of the connected components of $\boldsymbol{\mu}_3$ is not geometrically connected.

(3) Let $\mathbf{G}$ be a linear algebraic over an algebraically closed field k. Show that $\gamma_{\mathbf{G}} : \mathbf{G} \to \boldsymbol{\pi}_0(\mathbf{G})$ is a $\mathbf{G}^0$-torsor (namely $\mathbf{G}$ is a $\mathbf{G}^0$-torsor over $\boldsymbol{\pi}_0(\mathbf{G})$).

[3] More generally when $\mathbf{G}$ is a smooth affine group scheme defined over a base scheme $\mathbf{S}$, the connected component of $\mathbf{G}$ is the open group subscheme of $\mathbf{G}$ whose underlying topological set is the union $\cup_{s \in S} \mathbf{G}_s^0$