LCA's

least - common - Ancestors (LCA's)

Directed Acyclic Graph (DAG)

is directed graph $G=(V,E)$ s.t. $\nexists$ $v_1 \dots v_k$, $k>1$ with $v_1 \to v_2 \to v_3 \dashrightarrow v_k \to v_1$

DAG's have: **roots** (vertices with indeg 0) // $R(G)$

leaves, also called sinks (vertices with outdeg 0) // $L(G)$

A DAG is a **network** if it has exactly 1 root.

In biology/bioinf. etc it is often assume that DAGs are **phylogenetic**: no v with $\text{indeg}(v) = \text{outdeg}(v) = 1$.

not phylo: 

why?

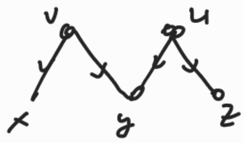
"no historical trace left, that justifies this vertex."

Can this be generalized?

yes: LCA's.

For a DAG G & $A \subseteq L(G)$, vertex $v \in V(G)$ is a common ancestor of A if v is an ancestor of all $x \in A$.

$ANC(A)$ = set of all ancestors of A



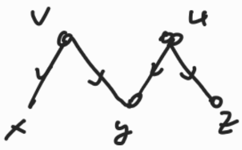
$ANC(\{x,y\}) = \{v\}$

$ANC(\{x,y,z\}) = \emptyset$

v is a least common ancestor if v is a $\leq_G$-minimal vertex in $ANC(A)$.

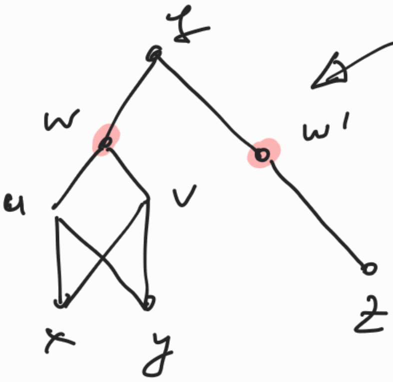
$LCA(A)$ = set of all least common ancestors

If $LCA(A) = \{v\}$, then we write $v = lca(A)$



$v = lca(\{x,y\})$

$LCA(\{x,z\}) = \emptyset$



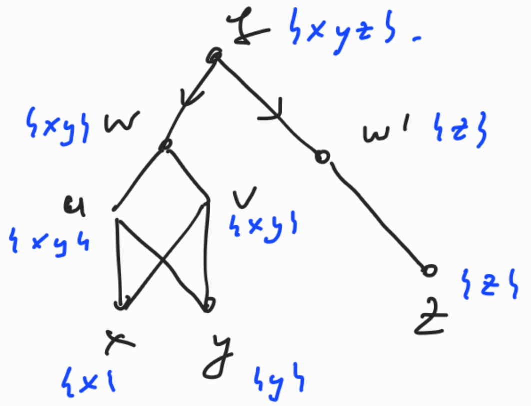
phylo vs LCA's

Q: · when is v LCA / unique LCA of some $A \subseteq L(G)$?

· when are all vertices in DAG LCA's/unique LCA's of some $A \subseteq L(G)$?

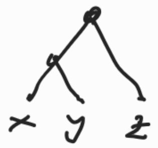
Cluster in DAG (G): $C(v) = \{x \in L(G) : x \preceq_G v\}$

$\mathfrak{C}_G$ = set of all clusters in G.



$\mathfrak{C}_G = \{\{x\}, \{y\}, \{z\}, \{xy\}, \{xyz\}\}$

Note T:



here $\mathfrak{C}_T = \mathfrak{C}_G$

Obs 1 if $v \preceq_G u \Rightarrow C_G(v) \subseteq C_G(u)$

L.1

DAG $G=(V,E)$, $v \in V$, $A \subseteq L(G)$ non-empty
the following are equivalent:

(1) $v \in LCA(A)$

(2) $A \subseteq C_G(v)$, $A \not\subseteq C_G(u)$ $\forall (vu) \in E$

(3) $A \subseteq C_G(v)$, $A \not\subseteq C_G(u)$ $\forall u \in V, u \prec_G v$

In particular, $v \in LCA(A) \Rightarrow C_G(u) \neq C_G(v)$ $\forall (vu) \in E$

proof: (1)⇒(3) by def of LCA.
(3)⇒(2) trivial.

(2)⇒(1) $A \subseteq C(v) \Rightarrow v \in ANC(A)$

$A \not\subseteq C(u)$ $\forall (vu) \in E$ & Obs 1 $\Rightarrow A \not\subseteq C(w)$ $\forall w \prec_G v$

$\Rightarrow v$ is $\preceq_G$-min vertex with $A \subseteq C(v) \Rightarrow v \in LCA(A)$

⇒ 1, 2, 3 equiv.

Now assume $v \in LCA(A) \overset{(2)}{\Longrightarrow} A \not\subseteq C(u) \ \forall (vu) \in E$
$\& \ A \subseteq C(v)$

$\Rightarrow \ C(v) \neq C(u) \ \forall (vu) \in E.$

$\square$

L.2 For DAG $G = (V, E)$, $v \in V$ following equivalent:

(1) $v \notin LCA(A)$ for any non-empty $A \subseteq L(G)$

(2) $\exists (vu) \in E$ st $C_G(u) = C_G(v)$

(3) $v \notin LCA(C_G(v))$

proof: (1): $\Rightarrow v \notin LCA(C_G(v))$ (1 ⇒ 3, but go 1 ⇒ 2 ⇒ 3 ⇒ 1)

$\overset{L1}{\Longrightarrow} \exists (vu) \in E$ st $C_G(v) \subseteq C_G(u)$

$\& \ u < v \overset{Obs!}{\Longrightarrow} C(u) \subseteq C(v) \Rightarrow C(v) = C(u) \Rightarrow (2)$

(2) Assume $\exists (vu) \in E$ st $C(u) = C(v)$
Since $u \prec_G v \overset{Def.}{\Longrightarrow} v \notin LCA(C(v)) \Rightarrow (3)$

(3) $v \notin LCA(C(v))$

* Suppose, for contradiction $v \in LCA(A)$ for some A
By L1, $A \subseteq C(v)$ & $A \not\subseteq C(u) \ \forall \ u \prec_G v$

$\Longrightarrow$ but now for any A' with $A \subseteq A' \subseteq C(v)$
we have $A' \not\subseteq C(u)$

& thus, in particular, $A' = C(v)$
we have $C(v) \not\subseteq C(u)$

$\Rightarrow v \in LCA(C(v))$ ↯

$\Longrightarrow (1)$ $\square$

* implies:

Lemma 3   IF $v \in LCA(A) \Rightarrow v \in LCA(C(v))$.

L.4

For DAG $G=(V,E)$, $v\in V$ following equivalent:

(1) $v \neq lca(A)$ for any non-empty $A \in L(G)$

(2) $\exists (vu)\in E$ s.t. $C(u)=C_G(v)$ OR $|LCA(C(v))| \geq 2$

(3) $v \neq lca(C_G(v))$

proof: (1): IF $v \in LCA(C(v)) \Rightarrow |LCA(C(v))| \geq 2$

v not unique LCA

IF $v \notin LCA(C(v)) \overset{L2}{\Rightarrow}$ (2)

(2): IF $|LCA(C(v))| \geq 2 \Rightarrow v \neq lca(C(v))$

IF (vu), $C(u)=C(v) \overset{L2}{\Rightarrow} v \notin LCA(C(v))$

$\Rightarrow v \neq lca(C(v)) \Rightarrow$ (3)

(3): $v \neq lca(C(v))$

Suppose, for contradiction, $v = lca(A)$ for some A

Let A' be such that $A \subseteq A' \subseteq C(v)$

$\Rightarrow v \in ANC(A')$

$\Rightarrow \exists\; w \in LCA(A')$ with $w \leq_G v$

Since $A \subseteq A' \Rightarrow w \in ANC(A)$

Hence, if $w \neq v \Rightarrow w <_G v \Rightarrow$ contrad. to $v = lca(A)$

$\Rightarrow w = v$ holds., i.e. $v \in LCA(A')$

Suppose $|LCA(A')| > 1 \Rightarrow \exists\; u \in LCA(A')$, $u \neq v$.

Note u & v must be $\leq_G$-incomp. "$u \leq v$ lca"!

Since $A \subseteq A' \Rightarrow u \in ANC(A)$

This & $v = lca(A)$ implies $v \leq_G u$ ↯ $\Rightarrow LCA(A') = \{v\}$

$\Rightarrow v = lca(A') \ \forall A'$ with $A \subseteq A' \subseteq C(v)$

$\Rightarrow v = lca(C(v))$ ↯

$\Rightarrow v \neq lca(A) \ \forall A$. □

(*) implies:

Lemma 5 If $v = lca(A) \Rightarrow v = lca(C(v))$.

Contraposition of L2, L4 (1) & (3):

Cor 1 G DAG, $v \in V(G)$. Then it holds that:

$v \in LCA(A)$ for some $\emptyset \neq A \subseteq C(v) \iff v \in LCA(C(v))$

$v = lca(A)$ — " — $\iff v = lca(C(v))$

Def: A DAG $G=(V,E)$, is

- LCA-relevant (LCA-rel), if $\forall v \in V$: $v \in LCA(A)$ for some $A \subseteq L(G)$
- lca-relevant (lca-rel), if $\forall v \in V$: $v = lca(A)$ for some $A \subseteq L(G)$

Obs 2:
- lca-relevant $\Rightarrow$ LCA-relevant.
- lca-rel & LCA-rel. DAGs are phylogenetic.

( $outdeg(v) = 1 \Rightarrow C(v) = C(u)$)

Without proof: Finding vertices that are not lca/LCA vertices can be done in polytime.

Theorem 1: $G=(V,E)$ is LCA-rel $\iff \nexists\, (u,v)\in E: C_G(u)=C_G(v)$

(proof: apply L2 to all vertices)

Theorem 2: Following eqn. for all DAGs $G=(V,E)$

(1) G LCA-rel

(2) $v = \operatorname{lca}(C(v))\ \forall v\in V$ "strong cluster-lca property" (CL)

(3) $\operatorname{lca}(C(v))$ well-def $\forall v\in V$ " cluster-lca property" (CL)
& G is LCA-rel.

(4) G satisfies PCC: u & v are $\preceq$-comparable
$\iff C(u)\subseteq C(v)$ or $C(v)\subseteq C(u)\ \forall u,v\in V$
& G is LCA-rel

(5) G satisfies PCC
& $u\neq v \Rightarrow C(u)\neq C(v)\ \forall u,v\in V$

(6) $C(u)\subseteq C(v) \iff u\preceq v\ \forall u,v\in V$

proof: (1) $\iff$ (2) by COR 1

(3): Let $v\in V$.

G LCA-rel $\Rightarrow v\in LCA(A)$ for some A
$\overset{L3}{\Rightarrow} v\in LCA(C(v))$

Since $LCA(C(v))$ well-def $\Rightarrow v=\operatorname{lca}(C(v))\ \forall v$

Hence (3) $\Rightarrow$ (1)

more (1) & (2) $\Rightarrow$ (3) in summary (1) (2) (3) equiv.

Show now $(2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (3)$

$(2) \Rightarrow (4)$ By Obs 1: $u \leq v \Rightarrow C(u) \subseteq C(v) \quad \forall u,v$

Assume now that $C(u) \subseteq C(v)$.

By (2) $u = \mathrm{lca}(C(u))$

Since $C(u) \subseteq C(v)$ & Def of lca: $u \leq v$

$\Rightarrow$ G satisfies (PCC) ✓

Since $(2) \Leftrightarrow (3) \Rightarrow$ G LCA-rel.

$(4) \Rightarrow (5)$: Since G satisfies (PCC) $\Rightarrow \nexists$ $\leq$-incomp vertices with $C(u) = C(v)$

$\Rightarrow$ if u & v are $\leq$ incomp $\Rightarrow C(u) \neq C(v)$.

Since G LCA-rel $\overset{\text{Thm 1}}{\Rightarrow} \nexists (v,u) \in E : C(u) \neq C(v)$

This + Obs 1: $u < v \Rightarrow C(u) \subsetneq C(v)$

$\Rightarrow \forall$ dist u,v: $C(u) \neq C(v)$.

$(5) \Rightarrow (6)$: Let $u,v \in V$. if $u \preceq v \overset{\text{Obs 1}}{\Rightarrow} C(u) \subseteq C(v)$.

Suppose now that $C(u) \subseteq C(v)$ $*$

By PCC: u & v are $\leq_G$-comparable

If $v < u$ then $u \neq v$ & $C(u) \neq C(v)$

$\overset{\text{Obs 1}}{\Rightarrow} C(v) \subsetneq C(u)$ ↯ $*$

$\Rightarrow u \leq v$

(6) $\Rightarrow$ (3) By conditions in (6), G satisfies PCC.
Let $v \in V$

$W := \{w : w \in V,\ C(w) = C(v)\}$.

By PCC the elements in W must be pairwise $\leq_G$-comp.
$\Rightarrow$ W contains a unique $\leq_G$-min. element w

By PCC, we also have $C(v) = C(w) \subsetneq C(u) \Rightarrow w <_G u$
(else $u \leq w \overset{\text{Obs.7}}{\Rightarrow} C(u) \subseteq C(w)$)

In summary: $w = \text{lca}(C(u))$ well-def. $\forall u \in V$

$\Rightarrow$ G satisfies (CL).

If v not LCA-vertex $\Rightarrow$ $v \notin \text{LCA}(A)\ \forall A$.
By L2, $\exists (vu) \in E$: $C(u) = C(v)$

But cond (6) implies that
$C(u) \subseteq C(v) \Rightarrow u \leq v$
$C(v) \subseteq C(u) \Rightarrow v \leq u$
$\Rightarrow u = v$ ↯

$\Rightarrow$ v LCA vertex $\forall v$, i.e., G LCA rel.

$\Rightarrow$ (3) holds

$\square$

[how Beamer slides $\rightarrow$ simplification]