Vi har visat

$\boxed{d=2}$

Sats (Greens sats)

Om $D \subset \Omega$ är en reguljärt $(\Omega \subset \mathbb{R}^2)$ område på planet.

$F \in \mathcal{C}^1(\Omega, \mathbb{R}^2) \subset \mathcal{C}^1(\Omega, \mathbb{R}^3)$

$$\int_{\partial D} F \cdot \vec{T}\, ds = \iint_D \underbrace{(\nabla \times F)\cdot e_3}_{\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)}\, dx dy$$

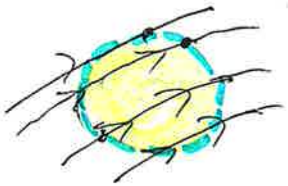
och

$$\int_{\partial D} F \cdot \vec{N}\, \underbrace{ds}_{\text{längd}} = \iint_D (\nabla \cdot F)\, dx dy.$$

där

$$\nabla \times F = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x} \end{pmatrix} \qquad F = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix}$$

$$\nabla \cdot F = \frac{\partial F^1}{\partial x_1} + \frac{\partial F^2}{\partial x_2}.$$



$\int_{\partial D} F \cdot \vec{N}\, ds \equiv$ Hur mycket flöd går utanför ∂D.

$\boxed{d=3}$

$\boxed{\text{Sats}}$ (Gauss satsen / divergens sats)

Om $K \subset \mathbb{R}^3$ är en kropp vars ∂K består av ändliga många slutna ytor, var och en orienterad så att normal enhetsvektor pekar ut från kroppen K. $K \subset \Omega \subset \mathbb{R}^3$

Det gäller, $F \in \mathcal{C}^1(\Omega, \mathbb{R}^3)$

$$\underbrace{\iint_{\partial K} F \cdot \vec{N}\, \underbrace{d\vec{S}}_{\text{Area element}}}_{\text{al}} = \underbrace{\iiint_K \nabla \cdot F\, dx dy dz}_{\text{Volym integral}}$$

[Flödet av vektorfältet ut genom sidorna av kroppen K]

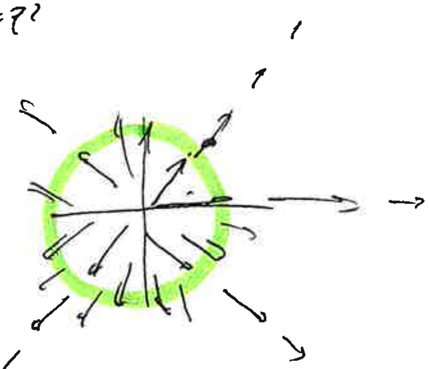
Exempel.

Låt $\left[F(x,y,z) = \dfrac{r}{|r|^3} \right]$

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$|r| = \sqrt{x^2+y^2+z^2}$

$F \in \mathcal{C}^1(R^3 \setminus \{\vec{0}\})$

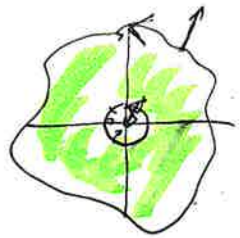


$$\iint_{\partial B_\varepsilon(\vec{0})} F \cdot n \, dS = \left[\vec{n} = \frac{r}{|r|} \right]$$

$$= \iint_{\partial B_\varepsilon(\vec{0})} \underbrace{\frac{r}{|r|^3} \cdot \frac{r}{|r|}}_{\frac{|r|^2}{|r|^4} = \frac{1}{|r|^2}} \, dS = \frac{1}{\varepsilon^2} \left[\underbrace{\iint_{\partial B_\varepsilon(\vec{0})} dS}_{4\pi\varepsilon^2} \right] = 4\pi$$

Om K är en kropp. som uppfyller villkore för att använda Gaussatsen.

Antag att $0 \in \mathring{K}$ ($\exists \varepsilon > 0 \; B(0,\varepsilon) \subset K$.)



Vi vill beräkna $\iint_{\partial K} F \cdot \vec{n} \, dS$

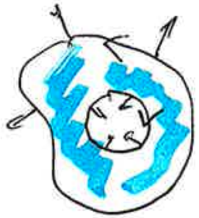
Let $K_\varepsilon = K \setminus B_\varepsilon(\vec{0})$

så gäller $\iint_{\partial K_\varepsilon} F \cdot n \, dS = \left[\begin{smallmatrix} \text{Gauss} \\ \text{sats} \end{smallmatrix} \right] = \iiint_{K_\varepsilon} (\nabla \cdot F) \, dx\,dy\,dz = 0$

$\boxed{\nabla \cdot F = 0}$ (uppgift)

$$\iint_{\partial K_\varepsilon} F \cdot n \, dS = \iint_{\partial K} F \cdot n \, dS - \iint_{\partial B_\varepsilon(0)} F \cdot n \, dS$$

n orientering pekar ut från $B_\varepsilon(0)$



$$\Rightarrow \iint_{\partial K} F \cdot n \, dS = \iint_{\partial B_\varepsilon(0)} F \cdot n \, dS = 4\pi$$

Vi har definierad divergens som den operator.

$$\mathrm{div}: \mathscr{C}^1(\Omega, \mathbb{R}^3) \longrightarrow \mathscr{C}(\Omega, \mathbb{R})$$

$$F \longmapsto \mathrm{div}\, F = \nabla \cdot F = \sum_{j=1}^{3} \frac{\partial F^j}{\partial x_j}$$

Vi definierar rotation (curl) som operationen

$$\mathrm{rot}: \mathscr{C}^1(\Omega, \mathbb{R}^3) \longrightarrow \mathscr{C}^0(\Omega, \mathbb{R}^3)$$

$$F \longmapsto \mathrm{rot}\, F = \nabla \times F = \begin{vmatrix} e_1 & \partial_1 & F^1 \\ e_2 & \partial_2 & F^2 \\ e_3 & \partial_3 & F^3 \end{vmatrix}$$

$$= e_1(\partial_2 F^3 - \partial_3 F^2) + e_2(\partial_3 F^1 - \partial_1 F^3) + e_3(\partial_1 F^2 - \partial_2 F^1)$$

$(1, 2, 3)$ $(2\ 3\ 1)$ $(3\ 1\ 2)$

reellvärda
u funktion

$$\nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \qquad \nabla u = \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \partial_3 u \end{pmatrix}$$

$$\nabla \cdot F = (\partial_1\ \partial_2\ \partial_3)\begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix}$$

$u \cdot (v \times w) = \det \begin{pmatrix} u & v & w \end{pmatrix}$

i) $\nabla \cdot (\nabla \times F) = 0$

ii) $\nabla \times (\nabla f) = \vec{0}$

iii) $\nabla \cdot (\nabla f) = (\nabla \cdot \nabla) f = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} f = \underbrace{\Delta f = \nabla^2 f}_{\text{Laplacian av } f.}$

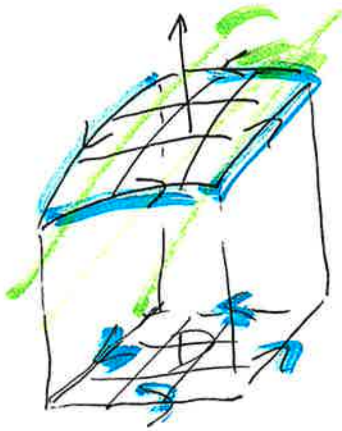
iv) $\nabla \cdot (F \times G) = \cancel{\begin{vmatrix} \partial_1 & F^1 & G^1 \\ \partial_2 & F^2 & G^2 \\ \partial_3 & F^3 & G^3 \end{vmatrix}}$

$$\left[\nabla \cdot (F \times G) = (\nabla \times F) \cdot G + (\nabla \times G) \cdot F . \right]$$

Sats (Stokes sats)

Antag att Γ är en orienterad yta i rummet med positivt orienterad rand.

Om $F \in \mathcal{C}^2(\Omega, \mathbb{R}^3)$ så gäller



$$\int_{\partial\Gamma} F \cdot dr = \iint_\Gamma (\nabla \times F) \cdot n \, dS$$

$$\Gamma = \mathrm{Graf}(g) = \{(s,t, g(s,t)) \;, (s,t) \in D\} =$$

$$\vec{n} = \frac{\frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t}}{\left|\frac{\partial g}{\partial s} \times \frac{\partial g}{\partial s}\right|} = \frac{1}{\sqrt{1 + |\nabla g(\cdot)|^2}} \begin{pmatrix} -\frac{\partial g}{\partial s} \\ -\frac{\partial g}{\partial t} \\ 1 \end{pmatrix}$$

$\partial\Gamma = \mathcal{C}(\partial D)$

Om $u \mapsto \underbrace{(s(u), t(u))}_{r(u)}$ $[a,b]$. parametrisering av randen av D.

Så $u \mapsto (s(u), t(u), g(s(u), t(u))) = (r(u), g(r(u)))$ $= \gamma(u)$ parametriserar $\partial\Gamma$

"Bevis" $\int_{\partial\Gamma} F \cdot dr$

$$= \int_{\partial\Gamma} F^1 dx_1 + F^2 dx_2 + F^3 dx_3$$

$$\gamma(u) = \begin{pmatrix} s(u) \\ t(u) \\ g(s(u), t(u)) \end{pmatrix} \qquad \dot\gamma(u) = \begin{pmatrix} \dot s(u) \\ \dot t(u) \\ \partial_s g(s(u),t(u)) \cdot \dot s(u) + \partial_t g(s(u),t(u))\, \dot t(u) \end{pmatrix}$$

$$\underbrace{\int_{\partial P} F \cdot dr}_{\text{Bor i } \mathbb{R}^3} = \int_a^b F^1(\gamma(u))\, \dot s(u) + F^2(\gamma(u))\, \dot t(u) + F^3(\gamma(u))\left(\partial_s g(r(u))\, \dot s(u) + \partial_t g(r(u)) \cdot \dot t(u)\right) du$$

$$= \int_a^b \Big[\left(F^1(\gamma(u)) + F^3(\gamma(u))(\partial_s g)(r(u))\right) \dot s(u) +$$
$$F^2(\gamma(u)) + F^3(\gamma(u))(\partial_t g)(r(u))\Big)\, \dot t(u) \Big]\, du.$$

$$\bar F(s,t) = F(s, t, g(s,t))$$

$$= \int_a^b \Big[\left(\bar F^1(r(u)) + \bar F^3(r(u))(\partial_s g)(r(u))\right) \dot s(u)$$
$$+\left(\bar F^2(r(u)) + \bar F^3(r(u))(\partial_t g)(r(u))\right] \dot t(u) \Big]\, du.$$

$$= \underbrace{\int_{\partial D} \underbrace{\left(\bar F^1 + \bar F^3(\partial_s g)\right)}_{P} ds + \underbrace{\left(\bar F^2 + \bar F^3(\partial_t g)\right)}_{Q} dt}_{\text{Bor i } \mathbb{R}^2}.$$

$$= \left[\text{Green Sats}\right] = \iint_D \left(\frac{\partial Q}{\partial s} - \frac{\partial P}{\partial t}\right) ds\,dt$$

Notera at

$$F(s, t, g(s,t)) = \bar{F}(s,t)$$

Så

$$\frac{\partial Q}{\partial s} = \left(\frac{\partial F^2}{\partial x_1} + \frac{\partial F^2}{\partial x_3} \cdot \frac{\partial g}{\partial s}\right) + \left(\frac{\partial F^3}{\partial x_1} + \frac{\partial F^3}{\partial x_3} \frac{\partial g}{\partial s}\right)\frac{\partial g}{\partial t} + F^3 \frac{\partial^2 g}{\partial s \partial t}$$

$$\frac{\partial P}{\partial t} = \left(\frac{\partial F^1}{\partial x_2} + \frac{\partial F^1}{\partial x_3} \frac{\partial g}{\partial t}\right) + \left(\frac{\partial F^3}{\partial x_2} + \frac{\partial F^3}{\partial x_3} \cdot \frac{\partial g}{\partial t}\right)\frac{\partial g}{\partial s} + F^3 \frac{\partial^2 g}{\partial t \partial s}$$

$$= \left(\frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x_2}\right) + \left(\frac{\partial F^3}{\partial x_2} - \frac{\partial F^2}{\partial x_3}\right)\left(-\frac{\partial g}{\partial s}\right) + \left(\frac{\partial F^1}{\partial x_3} - \frac{\partial F^3}{\partial x_1}\right)\left(-\frac{\partial g}{\partial t}\right)$$

$$= \left(-\frac{\partial g}{\partial s}, -\frac{\partial g}{\partial t}, 1\right) \begin{pmatrix} \frac{\partial F^3}{\partial x_2} - \frac{\partial F^2}{\partial x_3} \\ -\left(\frac{\partial F^3}{\partial x_1} - \frac{\partial F^1}{\partial x_3}\right) \\ \frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x_2} \end{pmatrix}$$

$$= \sqrt{1 + |\nabla g|^2}\,(\vec{n} \cdot \nabla \times F)(s, t, g(s,t))$$

Därför

$$\iint_D \left(\frac{\partial Q}{\partial s} - \frac{\partial P}{\partial t}\right)(r(\quad ds\,dt$$

$$= \iint_D (\vec{n} \cdot \nabla \times F)(s,t,g(s,t)) \cdot \sqrt{1+|\nabla g|^2}\, ds\,dt$$

$$= \iint_\Gamma (\nabla \times F) \cdot n \, dS$$