MH5011-F10

Vi har visat.

$\boxed{d=3}$

Sats (Stokes)

Γ orienterad yta i rummet ($\mathcal{C}^2$ ytan) med positivt orienterade rand $\partial\Gamma$

F vectorfält $\mathcal{C}^1$

Då gäller

$$\int_{\partial\Gamma} F\cdot\vec{T}\, ds = \iint_{\Gamma} (\nabla\times F)\cdot n\, dS$$

$\Gamma = \text{graf}(g)$

Sats (Gauss / divergens)

Om K är en kropp vars ∂K består av ändligt många slutna ytor, var och en orienterad så att normal enhetsvektorfält pekar ut från kroppen K.

$$\iint_{\partial K} F\cdot\vec{n}\, dS = \iiint_K \nabla\cdot F\, dx\,dy\,dz$$

$\boxed{d=2}$

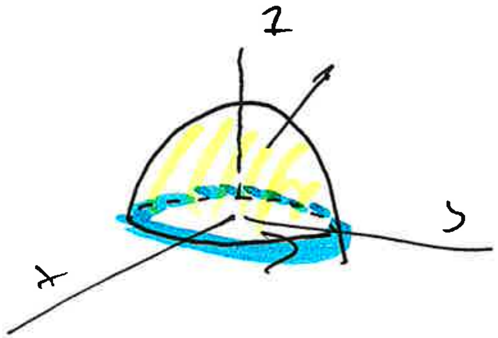
Om D är en reguljärt området på planet sådan att ∂D är positivt orienterad.

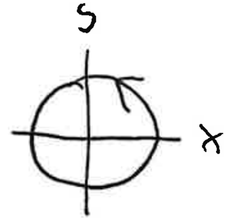
$$\int_{\partial D} F\cdot\vec{T}\, ds = \iint_D (\nabla\times F)\cdot e_3\, dx\,dy. \Big/ \int_{\partial D} F\cdot\vec{n}\, ds = \iint_D (\nabla\cdot F)\, dx\,dy.$$

Exempel.

$Y = \{(x,y,z) : x^2+y^2+z^2 = 9,\ z \geq 0\}.$

$\partial Y = \{(x,y,z) : z=0 \quad x^2+y^2 = 9\}.$





$$F = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

$$\int_{\partial Y} F \cdot ds = \iint_Y (\nabla \times F) \cdot n \, dS = I$$

$$\nabla \times F = \dots = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \qquad Y = \{(x,y,z) : \ z = (9 - x^2 + y^2)^{\frac{1}{2}},\ \underbrace{x^2+y^2 \leq 9}_{D = \{(x,y) : x^2+y^2 \leq 9\}}\}$$

$$\varphi(x,y) = (x, y, (9-x^2-y^2)^{\frac{1}{2}})$$

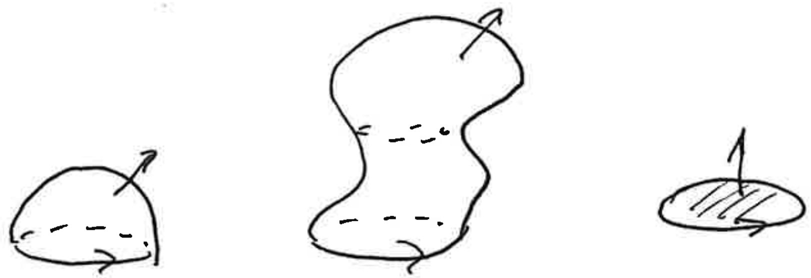
$$\frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = \dots = \begin{pmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \\ 1 \end{pmatrix}$$

$$\vec{n} = \frac{1}{\sqrt{1+|\nabla z|^2}} \begin{pmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \\ 1 \end{pmatrix}$$

$$I = \iint_D (\nabla \times F)(\varphi(x,y)) \cdot \vec{n}(\varphi(x,y)) \underbrace{\left| \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} \right|}_{\sqrt{1+|\nabla z|^2}} dx\,dy.$$

$$= \iint_{x^2+y^2 \le 9} \underbrace{(0,0,-2) \begin{pmatrix} -\partial z/\partial x \\ -\frac{\partial z}{\partial y} \\ 1 \end{pmatrix}}_{-2} dx\,dy. = -2 \underbrace{\iint_{x^2+y^2\le 9} 1 \; dx\,dy}_{\pi 3^2 \quad (\pi r^2)} = -28\pi$$

I det fall kunde vi har beräkna integralen direkt.



Konservativa vektorfältar

Vi visade (i dimension 2, men argument a lika för alla $d \geq 2$)

$F \in \mathscr{C}^1(\Omega, \mathbb{R}^d)$

$$F = \begin{pmatrix} F^1 \\ \vdots \\ F^d \end{pmatrix} = \sum F^j e_j \text{ är konservativt}$$

$$\Leftrightarrow \omega_F = \sum F^j dx_j \text{ är exakt}$$

$$\Leftrightarrow \exists u \quad du = \sum \frac{\partial u}{\partial x_j} dx_j = \omega_F$$

($u \in \mathscr{C}^2$, potentialt)

$$\Leftrightarrow \exists u : \nabla u = F$$

$\int_\gamma F \cdot ds$

$$\Leftrightarrow \forall \gamma \subset \Omega \text{ sluten och enkel kurva gäller } \int_\gamma \omega_F = 0$$

$\Leftrightarrow$

Ω sammanhängande

$$1 \leq j < k \leq d \quad \frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k} = 0$$

$$\Rightarrow u \in \mathscr{C}^2 \Rightarrow F^j = \frac{\partial u}{\partial x_j} \Rightarrow \frac{\partial F^j}{\partial x_k} = \frac{\partial^2 u}{\partial x_k \partial x_j} \overset{u \in \mathscr{C}^2}{=} \frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial F^k}{\partial x_j}$$

$$\Leftrightarrow \frac{\partial F^j}{\partial x_k} - \frac{\partial F^k}{\partial x_j} = 0$$

$\boxed{\Leftarrow}$ Om $d = 2$ Greens sats + Enkel sammanhängande området.

$\boxed{\Leftarrow}$ $(d=3)$

Om $\dfrac{\partial F^k}{\partial x_j} - \dfrac{\partial F^j}{\partial x_k} = 0 \qquad 1 \le j < k \le 3.$

$$\begin{cases} \dfrac{\partial F^3}{\partial x_1} - \dfrac{\partial F^1}{\partial x_3} = 0 \\[2ex] \dfrac{\partial F^3}{\partial x_2} - \dfrac{\partial F^2}{\partial x_3} = 0 \\[2ex] \dfrac{\partial F^2}{\partial x_1} - \dfrac{\partial F^1}{\partial x_2} = 0 \end{cases} \iff \nabla \times F = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = ($$

$$\begin{aligned} \nabla \times F = {} & \left(\frac{\partial F^3}{\partial x_2} - \frac{\partial F^2}{\partial x_3} \right) \vec{e}_1 && (1, 2, 3) \\ & + \left(\frac{\partial F^1}{\partial x_3} - \frac{\partial F^3}{\partial x_1} \right) \vec{e}_2 && (2, 3, 1) \\ & \left(\frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x_2} \right) \vec{e}_3 && (3, 1, 2) \end{aligned}$$

Def Vi säger att F är __virvelfri__ i Ω.

$$\forall x \in \Omega \quad (\nabla \times F)(x) = \vec{0}$$

Givet $S \subset \mathbb{R}^3$ yta (orienterad)

$$\int_{\partial S} F \cdot ds = \iint_S (\nabla \times F) \cdot \vec{n} \, dS \overset{\substack{F \text{ virvelfri} \\ \downarrow}}{=} 0$$

Sats. (Seiferts sats.)

Om $\gamma \subset \mathbb{R}^3$ är en orienterad sluten kurva, kan man hitta en orienterbar yta vars rand är γ.

($\gamma \subset \Omega$ $e\Omega$, enkel sammanhängande området /

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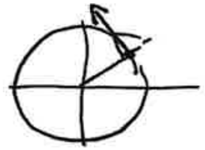
Låt $z(t) = x(t) + iy(t)$ vara en kurva orienterad (reguljärt) i planet



Ex: $z(t) = \cos t + i \sin t \longleftrightarrow \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$=: e^{it}$

$t \in [-\pi, \pi)$

$\dot{z}(t) = -\sin t + i \cos t = i\, z(t)$

Låt

$f(z) = u(z) + i v(z)$, u, v reellvärda.

vara en komplexvärd funktion.

$\left[\begin{array}{l} u : \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto u(x,y) \end{array} \right| \leadsto \left. \begin{array}{l} \tilde{u} : \mathbb{C} \longrightarrow \mathbb{R} \\ z = x+iy \longmapsto \tilde{u}(z) := u(x,y) \end{array} \right]$

$\boxed{u(z) \quad u(x,y)}$

Vi definera:

$$\int_\gamma f(z)\,dz := \left[\begin{array}{l} z(t): [a,b] \to \mathbb{C} \\ \quad t \mapsto z(t) = x(t) + i y(t) \end{array} \right]$$

$$= \int_a^b f(z(t)) \cdot \dot{z}(t)\, dt$$

komplext multiplikation.

$$\left.\begin{array}{l} \dot{z} = \dot{x} + i\dot{y} \\ f(z) = u + iv \end{array}\right] \to f(z(t))\,\dot{z}(t) = (u\cdot\dot{x} - v\dot{y}) + i(u\dot{y} + v\dot{x})$$

$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i\int_a^b (u\dot{y} + v\dot{x})\,dt$$

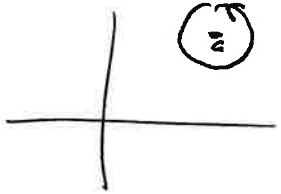
$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

Om vi definierar

$$[dz := dx + i\,dy]$$

Exempel.

Antag $\gamma = \partial B_r(a)$ positivt orienterad



$$\int_\gamma \frac{dz}{z-a} =$$

$$= \int_{-\pi}^{\pi} \frac{1}{z(t)-a} \cdot \dot{z}(t)\, dt$$

$$= \int_{-\pi}^{\pi} \frac{\cancel{ire^{it}}}{\cancel{re^{it}}}\, dt = 2\pi i$$

$z(t) = a + r\underbrace{e^{it}}_{\cos t + i \sin t}$

$\dot{z}(t) = ire^{it}$

$t \in [-\pi, \pi)$

Om $a = 0$

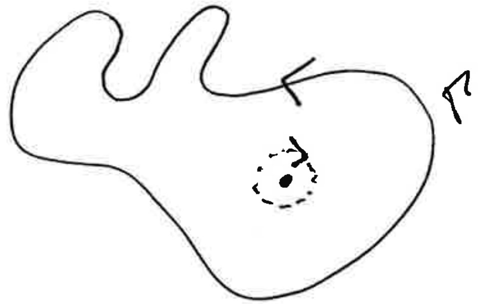
$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}.$$

Så $\int_\gamma \frac{1}{z}\, dz = \int_\gamma \left(\frac{x-iy}{x^2+y^2}\right)(dx + i\,dy)$

$\left(\begin{array}{l} \bar{z} = x - iy \\ z = x + iy \end{array}\right)$

$$= \underbrace{\int_\gamma \frac{x}{x^2+y^2}\, dx + \frac{y\, dy}{x^2+y^2}}_{= 0} + i \underbrace{\int_\gamma \frac{-y}{x^2+y^2}\, dx + x\frac{dy}{x^2+y^2}}_{2\pi}$$

Om Γ är en sluten, enkel kurva sådan att $0 \in \operatorname{Int}\Gamma$



$$\int_\Gamma \underbrace{\frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy}_{\omega} = \int_{\partial B_\varepsilon(0)} \omega_1 = 0$$

Greens sats

$$\int_\Gamma \underbrace{-\frac{y}{x^2+y^2}\,dx + x\,\frac{dy}{x^2+y^2}}_{\omega_2} = \int_{\partial B_\varepsilon(0)} \omega_2 = 2\pi$$

$$\Rightarrow \int_\Gamma \frac{1}{z}\,dz = 2\pi i \quad \text{om} \quad 0 \in \operatorname{Int}(\Gamma)$$

$$\left(\int_\Gamma \frac{1}{z-a}\,dz = 2\pi i \quad \text{om} \quad a \in \operatorname{Int}(\Gamma) \right)$$

$$\Leftrightarrow \boxed{\frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a}\,dz = 1}$$

Om $a \in \left(\overline{\operatorname{Int}\Gamma}\right)^c$

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{z-a}\,dz = 0$$

Antag att $u, v \in \mathcal{C}^1(\Omega)$, Ω enkel ~~område kopp~~ sammanhängande området

$f(z) = u(z) + i\,v(z)$, $\gamma \subset \Omega$ sluten enkel, positivt orienterad kurva.

$$\int_\gamma f(z)\,\underbrace{dz}_{dx+idy} = \int_\gamma (u\,dx - v\,dy) + i \int_\gamma u\,dy + v\,dx$$

$$\underset{\substack{\text{Greens}\\ \text{sats}}}{=} \iint_{\mathrm{Int}\,\gamma} \left[-\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] dx\,dy.$$

Om u, v uppfyller

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\[2ex] \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0 \end{cases} \quad \forall (x,y) \in \Omega. \implies \int_\gamma f(z)\,dz = 0$$

(Cauchy-Riemanns ekvationssystem)

$$\left[\begin{array}{l} \int_\gamma u\,dx - v\,dy = 0 \\ \int_\gamma u\,dy + v\,dx = 0 \end{array}\right]$$

$\forall \gamma \subset \Omega$

Ω är enkel sammanhängande.

Så $\exists\, U, V \in \mathcal{C}^1(\Omega)$, sådan att

$$\cancel{dU} \; dU = u\,dx - v\,dy \qquad\qquad \frac{\partial U}{\partial x} = u \quad \frac{\partial U}{\partial y} = -v$$
$$dV = v\,dx + u\,dy \quad \longleftrightarrow \quad \frac{\partial V}{\partial x} = v \quad \frac{\partial V}{\partial y} = u$$

Om vi definerar

$$F(z) = U(z) + i\,V(z)$$

Så gäller

$$\frac{\partial F}{\partial x} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv :=$$

$$\frac{\partial F}{\partial y} = \frac{\partial U}{\partial y} + i\frac{\partial V}{\partial y} = -v + iu = i\,\frac{\partial F}{\partial x}.$$

Så gäller

$$\boxed{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\, i = 0 \qquad \forall (x,y) \in D.}$$

Sats Antag att Ω är öppen, enkel sammanhängande mängd.

Om $u, v \in \mathcal{C}^1(\Omega)$ sådana att dem uppfyller

C-R-system d.v.s. $\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0 \end{cases}$ $\forall (x,y) \in \Omega.$

$f = u + iv$

$$\left[\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \overset{(CR)}{=} \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial \cancel{x}y} = -i \frac{\partial f}{\partial y} \right]$$

$$(CR) \Leftrightarrow \left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0 \right]$$

i) $\forall \gamma \subset \Omega$ sluten enkel styckvist reguljärt kurva gäller

$$\int_\gamma f(z)\, dz = 0$$

ii) $\exists F \in \mathcal{C}^2(\Omega | \mathbb{C})$ sådan att

i) $\dfrac{\partial F}{\partial x} + i \dfrac{\partial F}{\partial y} = 0$

ii) $\dfrac{1}{2}\left(\dfrac{\partial F}{\partial x} - i \dfrac{\partial F}{\partial y} \right) = f(z)$

$$\boxed{\begin{array}{l} \text{Wirtinger operatorer} \\ \dfrac{\partial}{\partial \bar{z}} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i \dfrac{\partial}{\partial y} \right) \\ \dfrac{\partial}{\partial z} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i \dfrac{\partial}{\partial y} \right) \end{array}}$$