Mn 5010 - F11

Sats

Om Ω är öppet, enkel sammanhängande, $u(x,y) = u(z)$

$f(z) = u(z) + i\,v(z)$, med $u, v \in \mathcal{C}^1(\Omega)$

som uppfyller Cauchy-Riemann ekvationssystem

$$(CR) \quad \begin{cases} \partial_x u - \partial_y v = 0 \\ \partial_y u + \partial_x v = 0 \end{cases} \quad \forall (x,y) \in \Omega \iff \left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0 \quad \forall (x,y) \in \Omega. \right]$$

Så gäller

i) $\forall \gamma \subset \Omega$ sluten, enkel styckvis reguljärt kurva gäller

$$\int_\gamma f(z)\, dz = 0$$

$dz = dx + i\,dy$.

ii) $\exists F \in \mathcal{C}^2(\Omega, \mathbb{C})$ sådan att

i) $\dfrac{\partial F}{\partial x} + i \dfrac{\partial F}{\partial y} = 0 \quad \forall (x,y) \in \Omega$ (F uppfyller (CR))

ii) $\dfrac{\partial F}{\partial x} - i \dfrac{\partial F}{\partial y} = 2 f(z) \quad \forall z \in \Omega.$

$f = u + iv$. $\quad (x+h)+iy$.

$$\frac{\partial f}{\partial x}(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\left(\frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{v(x+h,y)-v(x,y)}{h}\right)$$

$$= \frac{\partial u}{\partial x}(z) + i\,\frac{\partial v}{\partial x}(z)$$

$$\frac{\partial f}{\partial y}(z) = \frac{\partial u}{\partial y}(z) + i\,\frac{\partial v}{\partial y}(z)$$

Om u, v uppfyller (CR)

$$\frac{\partial f}{\partial x}(z) = \frac{\partial v}{\partial y}(z) + i\left(-\frac{\partial u}{\partial y}(z)\right) = -i\left[\frac{\partial u}{\partial y}(z) + i\,\frac{\partial v}{\partial y}(z)\right]$$

$$= -i\,\frac{\partial f}{\partial y}(z) \qquad \forall z\in\Omega.$$

$$\Leftrightarrow \qquad \frac{\partial f}{\partial x}(z) + i\,\frac{\partial f}{\partial y}(z) = 0$$

$$\Leftrightarrow \qquad \frac{\partial f}{\partial \bar z}(z) = 0 \quad \text{där} \quad \frac{\partial f}{\partial \bar z}(z) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z) + i\,\frac{\partial f}{\partial y}(z)\right)$$

$\forall z\in\Omega$.

Def. En funktion $f = u + iv$ där $u, v \in \mathcal{C}^1(\Omega)$ kallas analytisk i Ω om och endast om f uppfyller Cauchy-Riemanns ekvationer

$$\frac{\partial f}{\partial \bar{z}}(z) = 0 \quad \forall z \in \Omega. \quad \left(\Leftrightarrow \begin{array}{l} \partial_x u - \partial_y v = 0 \\ \partial_y u + \partial_x v = 0 \end{array} \quad \forall z \in \Omega \right)$$

<u>Exemplar.</u>

$$f(z) := e^z = e^x \cdot e^{iy} := e^x(\cos y + i \sin y). \qquad z = x + iy.$$

Så

$$u(x,y) = e^x \cos y \in \mathcal{C}^1(\mathbb{C})$$
$$v(x,y) = e^x \sin y \in \mathcal{C}^1(\mathbb{C})$$

Notera att

$$\left.\begin{array}{l} \partial_x u(x,y) = e^x \cos y = \partial_y v(x,y) \\ \partial_y u(x,y) = -e^x \sin y = -\partial_x v(x,y) \end{array}\right\} \quad \forall (x,y) \in \mathbb{R}^2$$

Så $f(z)$ är analytiskt i $\mathbb{C}$

Vi definierar

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

Vi har visat att

$$\frac{\partial}{\partial \bar{z}} e^{z} := 0$$

Notera att

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$$

och

$$\frac{\partial f}{\partial z}(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = e^{z}$$

$$f(z) = e^{z} = e^{x}e^{iy}.$$

$$\frac{\partial f}{\partial x} = e^{z}$$

$$\frac{\partial f}{\partial y} = e^{x}\frac{\partial}{\partial y}\big(\underbrace{e^{iy}}_{=\cos y + i\sin y.}\big)$$

$$= e^{x}(-\sin y + i\cos y)$$

$$= i e^{x}(\cos y + i\sin y)$$

$$= i e^{x}e^{iy} = i e^{z}$$

- Uppgift

i) e^{iz} är analytisk i $\mathbb{C}$

ii) $\cos z := \dfrac{e^{iz}+e^{-iz}}{2}$. $\qquad \left(\cos y = \dfrac{e^{iy}+e^{-iy}}{2} \right)$

$\sin z := \dfrac{e^{iz}-e^{-iz}}{2i}$

- $f(z) = z = x+iy$. analytisk i $\mathbb{C}$

Visa att $\dfrac{\partial f}{\partial \bar{z}}(z) = 0$.

- $f(z) = \bar{z}$ är inte analytisk. $\qquad \dfrac{\partial \bar{z}}{\partial \bar{z}} = 1$

$= x - iy =: u + iv$.

Notera att $\partial_x u = 1 \neq \partial_y v$

Lemma. Om f, g är analytisk så är det $f \pm g$ och $f \cdot g$.

Bevis (Uppgift $f+g$)

Notera att

$$\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial f}{\partial x} \cdot g + f \cdot \frac{\partial g}{\partial x}.$$

$$\frac{\partial}{\partial y}(f \cdot g) = \frac{\partial f}{\partial y} g + f \cdot \frac{\partial g}{\partial y}.$$

$$\Rightarrow \frac{\partial}{\partial \bar{z}}(f \cdot g) = \frac{1}{2}\left(\frac{\partial(f \cdot g)}{\partial x} + i \frac{\partial(f \cdot g)}{\partial y}\right) = \overbrace{\frac{\partial f}{\partial \bar{z}}}^{0} \cdot g + f \overbrace{\frac{\partial g}{\partial \bar{z}}}^{0} = 0$$

Följdsats.

$\forall p \in \mathbb{C}[z]$. $p(z) := \sum_{k=0}^{N} a_n z^k$, för något $N \in \mathbb{N}$. $a_0, \ldots, a_N \in \mathbb{C}$

är analytiska i $\mathbb{C}$.

Definition Låt u, v två reellvärda funktioner definierad i Ω.

Låt $f(z) = u(z) + i v(z)$.

Vi säger att f är komplext differentierbar i punkten $z \in \Omega$.

om och endast om existerar gränsvärdet

$$\lim_{w \to 0} \frac{f(z+w) - f(z)}{w}.$$

I det fall, gränsvärdet är unik och betecknas det som

$$f'(z).$$

och kallas den komplexa derivata av f.

Sats. Givet $u, v \in \mathcal{C}^1(\Omega)$

f är komplext differentierbar i alla punkter $z \in \Omega$.

om och endast om f uppfyller (CR) i Ω.

dvs. f är analytiskt i Ω.

$\Rightarrow)$ Om $f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w} \Leftrightarrow \left[\forall \varepsilon > 0 \ \exists \delta > 0 \quad 0 < |w| < \delta \Rightarrow \left| \frac{f(z+w) - f(z)}{w} - f'(z) \right| < \varepsilon \right]$

Så gäller ($w = h \in \mathbb{R}$)

$$f'(z) = \lim_{h \to 0} \frac{f(\overset{(x+h)+iy}{\overline{z+h}}) - f(z)}{h} = \frac{\partial f}{\partial x}(z)$$

Men det gäller också ($w = ih$)

$$f'(z) = \lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih} = \frac{1}{i} \frac{\partial f}{\partial y}(z) = -i \frac{\partial f}{\partial y}(z)$$

Så gäller

$$0 = \frac{\partial f}{\partial x}(z) + i \frac{\partial f}{\partial y}(z) \Leftrightarrow \frac{\partial f}{\partial \bar{z}}(z) = 0.$$

$$\left[\lim_{h \to 0} \left(\frac{u(x+h,y) - u(x,y)}{h} + i \frac{v(x+h,y) - v(x,y)}{h} \right) = \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y) \right]$$

Notera också

$$\frac{\partial f}{\partial z}(z) = \frac{1}{2} \left(\frac{\partial f}{\partial x}(z) - i \frac{\partial f}{\partial y}(z) \right) = \frac{1}{2} \left(f'(z) + f'(z) \right) = \frac{\cancel{2}}{\cancel{2}} f'(z)$$

$\Leftarrow)$ $\Delta z := \Delta x + i\Delta y$.

$$u(\cancel{z}+\Delta z) - u(z) = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= \underbrace{\partial_x u(x,y)\Delta x + \partial_y u(x,y)\Delta y}_{(\nabla u)^T\binom{x}{y}\binom{\Delta x}{\Delta y}} + |\Delta x + i\Delta y|\, g_1((\Delta x, \Delta y)).$$

Då $|\Delta z| = (|\Delta x|^2 + |\Delta y|^2)^{\frac{1}{2}}$

$g_1(\Delta z) \to 0$ då $\Delta z \to 0$.

Vi kan göra det samma för v.

$$\lim_{|\Delta z|\to 0} \frac{u(z+\Delta z) - u(z) - (\nabla u)^T\binom{x}{y}\binom{\Delta x}{\Delta y}}{|\Delta z|} = 0$$

Så

$$f(z+\Delta z) - f(z) = (\partial_x u(z) + i\partial_x v(z))\Delta x + (\underbrace{\partial_y u(z)}_{-\partial_x v(z)} + i\underbrace{\partial_y v(z)}_{\partial_x u(z)})\Delta y.$$

$$+ |\Delta z|\big(\underbrace{g_1(\Delta z) + g_2(\Delta z)}_{g(\Delta z)}\big)$$

$$-\partial_x v(z) + i\,\partial_x u(z) = i\left(\partial_x u(z) + i\,\partial_x v(z)\right)$$

$$f(z+\Delta z) - f(z) = \underbrace{\left(\partial_x u(z) + i\,\partial_x v(z)\right)}_{\frac{\partial f}{\partial x}(z)}\underbrace{(\Delta x + i\Delta y)}_{\Delta z} + |\Delta z|\, g(\Delta z)$$

$$\Rightarrow \quad \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial f}{\partial x}(z) + g(\Delta z)\frac{|\Delta z|}{\Delta z} \xrightarrow[\Delta z \to 0]{} \frac{\partial f}{\partial x}(z)$$

Så existerar gränsvärdet

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial f}{\partial x}(z)$$

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$f = u + iv$, $u, v \in \mathcal{C}^1(\Omega)$ Ω öppet.
enkel sammanhängande.

f är komplext differentierbar i Ω. $\iff$ u, v uppfyller CR

$$\begin{cases} \partial_x u - \partial_y v = 0 \\ \partial_y u + \partial_x v = 0 \end{cases} \quad \forall (x,y) \in \Omega.$$

$\iff$ f uppfyller $\frac{\partial f}{\partial \bar z}(z) = 0 \quad \forall z \in \Omega$.

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$\Downarrow$

$\forall \gamma \subset \Omega$ slutna kurva. $\int_\gamma f(z)\,dz = 0$ — Cauchys Integral sats

$\Updownarrow$ $dz = dx + i\,dy$

$$\begin{cases} \int_\gamma u\,dx - v\,dy = 0 \\ \int_\gamma v\,dx + u\,dy = 0 \end{cases}$$
$\forall \gamma \subset \Omega$ slutna kurvor

$\xRightarrow{\text{Greens sats}}$

$$\begin{cases} -\partial_x v - \partial_y u = 0 \\ \partial_x u - \partial_y v = 0. \end{cases} \quad \forall (x,y) \in \Omega.$$

$\Uparrow$

$\iff$ $\begin{pmatrix} u \\ -v \end{pmatrix}, \begin{pmatrix} v \\ u \end{pmatrix}$ är konservativa vektorfält i Ω.

$\iff$ $\exists U, V \in \mathcal{C}^2(\Omega)$, $\nabla U = \begin{pmatrix} u \\ -v \end{pmatrix}$, $\nabla V = \begin{pmatrix} v \\ u \end{pmatrix}$

$\Updownarrow$

$$\begin{cases} \exists F = U + iV, \quad U, V \in \mathcal{C}^2(\Omega) \\ \frac{\partial F}{\partial x} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z) \\ \frac{\partial F}{\partial y} = \frac{\partial U}{\partial y} + i\frac{\partial V}{\partial y} = -v + iu = i f(z) \end{cases}$$

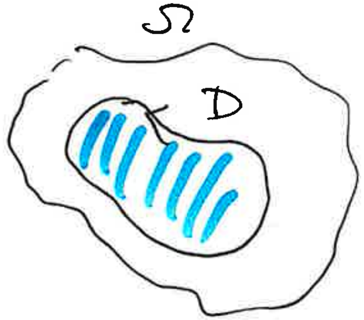
$\iff$

$$\begin{cases} \exists F = U + iV, \quad U, V \in \mathcal{C}^2 \\ \frac{\partial F}{\partial \bar z}(z) = 0 \\ \frac{\partial F}{\partial z}(z) = f(z) \end{cases}$$

Cauchy Integralsats.

Om $\Gamma \subset D$, Γ sluten kurva. och D är enkel sammanhängande

så gäller. $\int_\Gamma f(z)\,dz = 0.$



Cauchy Integralformeln.

Låt Γ vara en positivt orienterade randen till en öppet enkel sammanhängande området D.

Antas att f är analytisk i Ω där $\Omega \supset D \cup \Gamma$.

Så

$$\forall z \in D \quad f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta .$$

$\cancel{\left(\int_\Gamma \frac{1}{\zeta - z}\, d\zeta\right)}$

$D_\varepsilon = D \setminus \overline{B_\varepsilon(z)}$

$$\oint_{\partial D_\varepsilon} f(z)\,dz = \oint_{\partial D} f(z)\,dz - \oint_{\partial B_\varepsilon(z)} f(z)\,dz$$

$$\int_{\partial B_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, dz = \left[\begin{matrix} \zeta = z+\varepsilon e^{it} \\ t\in[0,2\pi]. \end{matrix}\right] =$$

$$= \int_0^{2\pi} \frac{f(\zeta(t))}{\zeta(t)-z} \cdot \dot\zeta(t)\, dt = \int_0^{2\pi} \frac{f(z+\varepsilon e^{it})}{\cancel{\varepsilon e^{it}}}\, i\, \cancel{\varepsilon e^{it}}$$

$$= i \int_0^{2\pi} f(z+\varepsilon e^{it})\, dt \xrightarrow[\varepsilon\to 0^+]{} i f(z)\, 2\pi$$

So

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z).$$

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$$\oint_{|z|=2} \frac{z}{z-i}\,dz = \oint_{|z|=2} \frac{f(z)}{z-i}\,dz = f(i)\,2\pi i$$

$$= -2\pi.$$

Cauchys integral formula.

$f(z) = z$