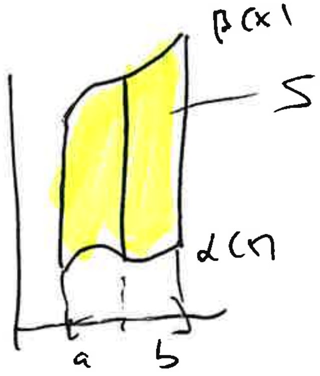
MM5011-F03

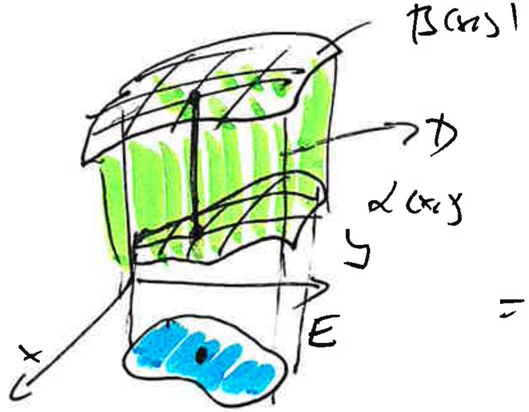
Beräkningar av flera variabler integrales. sker som dubbelintegrales med hjälp av itererad integration (Fubini's sats)

$$D = \{ (x,y,z) : \alpha(x,y) \le z \le \beta(x,y) ; (x,y) \in E \}.$$

$\alpha, \beta \in \mathcal{C}(E)$



$$\iint_S f(x,y)\,dxy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right) dx$$



$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_E \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dx\,dy.$$

Om vi kan uttrycka.

$$D = \{ (x,y,z) : x \in [a,b] ; (y,z) \in D_x \}.$$

Sådan att

$$D_x = \{ (y,z) : (x,y,z) \in D \}$$

Cilindriska koordinater.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$s > r > 0 \qquad \theta \in (-\pi, \pi)$

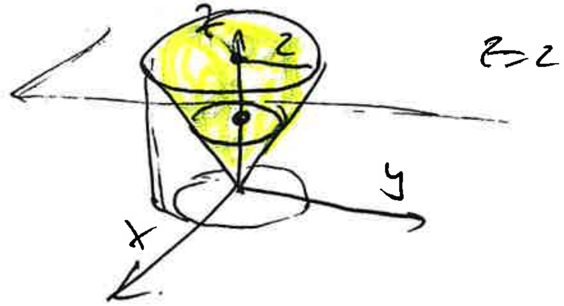
$\Omega_s = \{(r,\theta) : \ r \in (0,s),\ \theta \in (-\pi,\pi)\} \xrightarrow{\varphi} \mathbb{R}^2$

$\varphi(\Omega_s) = B_s(0) \setminus \{(x,y) : \cancel{y} \le 0\}.$

$$\varphi(r,\theta,z) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} \qquad \Omega_s = \underset{r}{(0,s)} \times \underset{\theta}{(-\pi,\pi)} \times \underset{z}{(-s,s)}.$$

$$d\varphi(r,\theta,z) = \left(\frac{\partial\varphi}{\partial r} \,\middle|\, \frac{\partial\varphi}{\partial\theta} \,\middle|\, \frac{\partial\varphi}{\partial z} \right) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(d\varphi(r,\theta,z)) = r > 0$$



$S = \{(x,y,z) : x^2+y^2 \leq 2z \quad , z \leq 2\}.$

$I = \iiint_S (x^2+y^2)\, dx\,dy\,dz$

$S = \{(x,y,z) : z \in [0,2] . : (x,y) \in \overline{B}_{\sqrt{2z}}(0)\}.$

$$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \int_0^2 \underbrace{\left(\iint_{x^2+y^2 \leq 2z} (x^2+y^2)\,dx\,dy \right)}_{I} dz.$$

$$I = \left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ \theta \in (-\pi,\pi) \\ r \in (0,\sqrt{2z}) \end{array} \right] = \int_0^{\sqrt{2z}} \int_{-\pi}^{\pi} r \cdot r^2 \, d\theta \, dr.$$

$$= 2\pi \int_0^{\sqrt{2z}} r^3 dr = 2\pi \frac{(2z)^2}{4} = 2\pi z^2$$

$\det(d\varphi(r,\theta)) = r.$

$$S = \{ (x,y,z) : \quad x^2+y^2 \leq 4 \; . \text{ och } \quad 2 \geq z \geq \frac{x^2+y^2}{2} \} .$$

$$I = \iint_{x^2+y^2 \leq 4} \left(\int_{\frac{x^2+y^2}{2}}^{2} (x^2+y^2) \, dz \right) dx \, dy .$$

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$$\iiint_S (x^2+y^2) \, dx \, dy \, dz = \left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \right.$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\iint_{D_x} f(x,y,z)\,dy\,dz \right) dx .$$

$$\iiint_{\varphi(\Omega)} f(\bar{x})\,d\bar{x} = \begin{bmatrix} \bar{x} = \varphi(y) \\ d\bar{x} = |\det d\varphi(y)|\,d\bar{y} \end{bmatrix} = \iiint_\Omega f(\varphi(\bar{y}))\,|\det d\varphi(\bar{y})|\,d\bar{y}$$

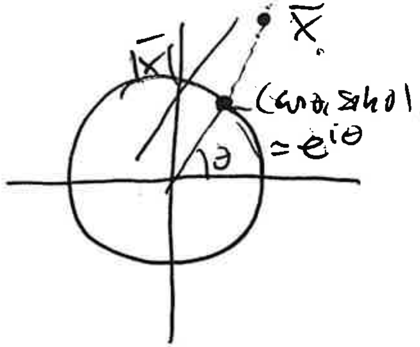
$\varphi : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ $\left[\begin{array}{llll} \Omega & \text{öppet} & \text{kvadrerbar} & \text{begränsad} \\ \varphi(\Omega) & \text{"} & \text{"} & \text{"} \end{array}\right]$

- $\varphi \in \mathcal{C}^1$
- φ injektivt ($\varphi(\Omega)$ ~~...~~ bijektivt med Ω).

$\det(d\varphi(\bar{x})) \neq 0 \quad \forall \bar{x} \in \Omega$.

$\varphi(\Omega) = \{ y : \exists x \in \Omega : \varphi(x) = y \}$.

Polära koordinater $d=3$.

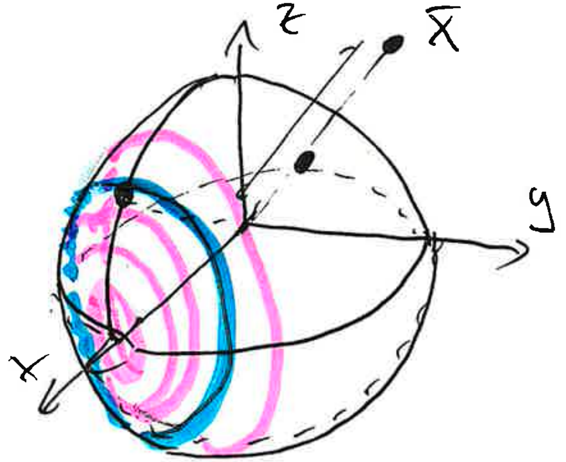


$$\begin{cases} x = |\bar{x}| \cos\theta \\ y = |\bar{x}| \sin\theta \end{cases}$$

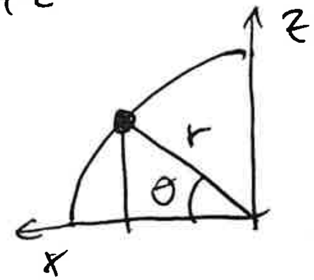
$$\iint_{B_s(0)} f(x,y)\,dx\,dy = \int_0^s \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$\Sigma_1 := \{(x,y) : x^2+y^2=1\}.$$

$$= \{(x,y) : x = \cos\theta,\ y = \sin\theta,\ \theta \in [-\pi, \pi)\}.$$



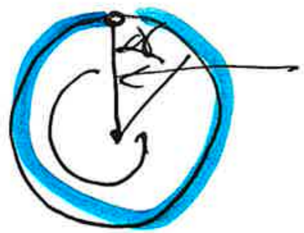
$$|\bar{x}| = \sqrt{x^2+y^2+z^2}$$



$$\begin{cases} x = r\cos\theta \\ y = (r\sin\theta)\cos\gamma \\ z = (r\sin\theta)\sin\gamma \end{cases}$$

$\theta \in [0, \pi]$

$\gamma \in [-\pi, \pi)$



$r\sin\theta$

$$\Sigma_2 = \{(x,y,z) : x^2+y^2+z^2=1\}.$$

$$= \{(r,\theta,\gamma) : x = r\cos\theta,\ y = r\sin\theta\cos\gamma,\ z = r\sin\theta\sin\gamma,\ \theta\in[0,\pi),\ \gamma\in[-\pi,\pi)\}.$$

$$\det\left(d\varphi_3(r,\theta,\gamma)\right) = r^2\cos\theta \det\left(\cos\theta\,\varphi_2(1,\gamma) \,\middle|\, \sin\theta \frac{\partial\varphi_2}{\partial\gamma}(1,\gamma)\right)$$

$$+ r^2\sin\theta \det\left(\sin\theta\,\varphi_2(1,\gamma) \,\middle|\, \sin\theta \frac{\partial\varphi_2}{\partial\gamma}(1,\gamma)\right)$$

$$= r^2\cos^2\theta \cdot \sin\theta \det\left(\varphi_2(1,\gamma) \,\middle|\, \frac{\partial\varphi_2}{\partial\gamma}(1,\gamma)\right)$$

$$+ r^2\sin^2\theta \sin\theta \underbrace{\det\left(\varphi_2(1,\gamma) \,\middle|\, \frac{\partial\varphi_2}{\partial\gamma}(1,\gamma)\right)}_{\det\left(d\varphi_2(r,\gamma)\right)\big|_{r=1}}$$

$$= r^2\sin\theta \underbrace{\det\left(d\varphi_2(r,\gamma)\right)\big|_{r=1}}_{=1}$$

$$\det(d\varphi_2(r,\gamma)) = \begin{vmatrix} \cos\gamma & -r\sin\gamma \\ \sin\gamma & r\cos\gamma \end{vmatrix} = r.$$

$$\boxed{\det\left(d\varphi_3(r,\theta,\gamma)\right) = r^2\sin\theta}$$

$\theta \in (0,\pi)$

$r \in (0,R)$

$\gamma \in (-\pi,\pi)$

$$\varphi_2(r,\gamma) = \begin{pmatrix} r\cos\gamma \\ r\sin\gamma \end{pmatrix} \qquad \Omega_S = (0,R) \times (-\pi,\pi)$$

$$\varphi_3(r,\theta,\gamma) = \begin{pmatrix} r\cos\theta \\ (r\sin\theta)\cos\gamma \\ (r\sin\theta)\sin\gamma \end{pmatrix} \qquad \Omega_S = \{(r,\theta,\gamma) : \ r\in(0,R),\ \theta\in[0,\pi),\ \gamma\in[-\pi,\pi]\}$$

$$= (0,R) \times (0,\pi) \times (-\pi,\pi)$$

$$= \begin{pmatrix} r\cos\theta \\ \varphi_2\underbrace{(r\sin\theta,\gamma)}_{r\sin\theta\, \varphi_2(1,\gamma)} \end{pmatrix}$$

$$d\varphi_3(r,\theta,\gamma) = \left(\frac{\partial\varphi_3}{\partial r} \,\middle|\, \frac{\partial\varphi_3}{\partial\theta} \,\middle|\, \frac{\partial\varphi_3}{\partial\gamma} \right) = \begin{array}{c} 1 \\ 2 \end{array}\begin{pmatrix} \cos\theta \\ \sin\theta\, \varphi_2(1,\gamma) \end{pmatrix}$$

$$= \begin{array}{c} 1 \\ 2 \end{array}\left(\begin{array}{c|c|c} \cos\theta & +\sin\theta & 0 \\ \sin\theta\, \varphi_2(1,\gamma) & r\cos\theta\, \varphi_2(1,\gamma) & r\sin\theta\, \dfrac{\partial\varphi_2}{\partial\gamma}(1,\gamma) \\ \underline{\qquad} & \underline{\qquad} & \\ 1 & 1 & \end{array} \right)$$

I fissa variabile $d \geq 4$

$$\varphi_d(r, \underset{(\theta_1, \bar{\theta}) \in \mathbb{R} \times \mathbb{R}^{d-1}}{\underset{\shortparallel}{\theta}}, \gamma) := \begin{pmatrix} r \cos \theta_1 \\ r \sin \theta_1 \, \varphi_{d-1}(1, \bar{\theta}, \gamma) \end{pmatrix}$$

$\underline{d=3}$

$x = r \cos \theta_1$

$y = r \sin \theta_1 \cos \gamma$

$z = r \sin \theta_1 \sin \gamma$

$\underline{d=4}$

$x_1 = r \cos \theta_1$

$x_2 = r \sin \theta_1 \cos \theta_1$

$x_3 = r \sin \theta_1 \sin \theta_2 \cos \gamma$

$x_4 = r \sin \theta_1 \sin \theta_2 \sin \gamma$.

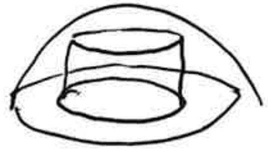
$$J_d(r, \theta, \gamma) := \det\left(d\varphi_d(r, \theta, \gamma)\right) = r^{d-1} \underbrace{\prod_{j=1}^{d-2} \sin^{(d-1)-j} \theta_j}_{\overset{!!}{=:}\, S_{d-1}(\theta)}$$

Vi vill beräkna. $I=\iiint_K 1\,dx\,dy\,dz$

$$K=\{(x,y,z) : x^2+y^2+z^2\le 9 \quad z\ge 0\}$$

$$=\{(x,y,z) : 0\le z\le 3 \quad x^2+y^2\le 9-z^2\}.$$

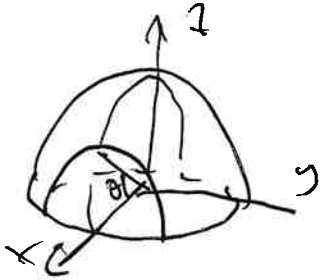
$$I=\int_0^3\left(\iint_{x^2+y^2\le 9-z^2} dx\,dy\right)dz = \int_0^3 \pi(9-z^2)\,dz = \dots$$



$$K=\{(x,y,z) : x^2+y^2\le 9 \quad 0\le z\le (9-(x^2+y^2))^{1/2}\}.$$

$$I=\iint_{x^2+y^2\le 9.}\left(\int_0^{(9-(x^2+y^2))^{1/2}} dz\right)dx\,dy=\dots$$

$$K=\{(x,y,z) : x^2+y^2+z^2\le 9 \quad z\ge 0\} \simeq \{(r,\theta,\gamma) : 0\le r\le 3, \ \theta\in[-\pi,\pi], \ \gamma\in[0,\pi]\}$$



$x=r\cos\theta$

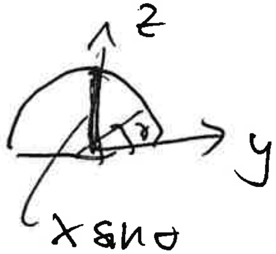
$y=r\sin\theta\cos\gamma$

$z=r\sin\theta\sin\gamma$

$\varphi(r,\theta,\gamma)$

$\theta\in(0,\pi)$

$\gamma\in(0,\pi)$



$$\det(\kappa(r,\theta,\gamma)) = r^2 \sin\theta .$$

$$\iiint_K dx\,dy\,dz = \iiint_{[0,3]\times[0,\pi]\times[0,\pi]} r^2 \sin\theta \, dr\, d\theta\, d\gamma . =$$

$$= \underbrace{\left(\int_0^3 r^2 dr\right)}_{\frac{3^3}{3}} \left(\int_0^\pi \sin\theta \, d\theta\right) \underbrace{\left(\int_0^\pi d\gamma\right)}_{\pi}$$

$$\underbrace{\left[-\cos\theta\right]_0^\pi}_{2}$$

$$= \frac{2\,3^3\,\pi}{3}$$

$$\mathbb{R}^2 = \Sigma_1 \times [0,+\infty)$$

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_0^{+\infty}\left(\int_{-\pi}^{\pi} r\, \cancel{df}\, f(r\cos\theta, r\sin\theta)\, d\theta\right) dr.$$

$$= \int_0^{\infty} r \left(\int_{\partial B_1(0)} f(r\omega)\, d\omega\right) dr$$

$$\mathbb{R}^3 = \Sigma_2 \times [0,\infty)$$

$$\iiint_{\mathbb{R}^3} f(x,y,z)\,dx\,dy\,dz = \int_0^{\infty} r^2 \underbrace{\left(\iint_{[0,\pi)\times[-\pi,\pi)} f(r,\theta,\gamma)\, \sin\theta \, d\theta\, d\gamma\right)}_{\iint_{\Sigma_2} f(r,\theta,\gamma)\, d\sigma(\theta,\gamma)} dr$$

$x = r\cos\theta$

$y = r\sin\theta\cos\gamma$

$z = r\sin\theta\sin\gamma$.