

**Time:** 09:00-14:00

**Instructions:**

- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Only **non-graphic** calculators are allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. JUSTIFY your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- Mark clearly where is your final answer putting A BOX around it.

**Grades:** Each solved problem is awarded by up to 10 points. At least 35 points are necessary for the grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

1. Calculate the limits

$$\text{a) } \lim_{x \rightarrow +\infty} \frac{1+x \ln x}{x+2 \ln x} \qquad \text{b) } \lim_{x \rightarrow 1} \frac{6x^2+4x-10}{9x^2-1}$$

**Solution** a) This is of indeterminate form  $\frac{\infty}{\infty}$ , so we apply L'Hôpital's rule to get

$$\lim_{x \rightarrow +\infty} \frac{1+x \ln x}{x+2 \ln x} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx}(1+x \ln x)}{\frac{d}{dx}(x+2 \ln x)} = \lim_{x \rightarrow +\infty} \frac{\ln x + x \frac{1}{x}}{1 + \frac{2}{x}} = \lim_{x \rightarrow +\infty} \frac{1 + \ln x}{1 + \frac{2}{x}} = \boxed{+\infty}$$

b) The function

$$\frac{6x^2+4x-10}{9x^2-1}$$

is continuous at 1, so

$$\lim_{x \rightarrow 1} \frac{6x^2+4x-10}{9x^2-1} = \frac{6+4-10}{9-1} = \boxed{0}$$

2. Consider the function  $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$ .

- (a) Find the critical points;
- (b) Find the intervals on which  $f$  is increasing and decreasing;
- (c) Is there any inflection point? Argument your answer.
- (d) Give the maximum and minimum value of  $f$  on the interval  $[0, 3]$ .

**Solution** (a) Observe that

$$f'(x) = x^2 + x - 6 = (x-2)(x+3) = 0 \Leftrightarrow x = 2 \quad \text{or} \quad x = -3.$$

So, the critical points are  $x = -3$  and  $x = 2$ .

(b) Using the relation between the sign of the derivative and the monotonicity of the function we have the situation reflected in the following table:

	$x < -3$	$x \in (-3, 2)$	$x > 2$
$f'(x)$	+	-	+
$f(x)$	↑	↓	↑

Than is, since  $f'(x) > 0$  for  $x < -3$  and  $x > 2$ , and  $f'(x) < 0$  for  $x \in (-3, 2)$  we have that  $f$  is increasing on  $(-\infty, -3) \cup (2, +\infty)$  and decreasing on  $(-3, 2)$ .

(c) Since  $f$  is continuous in a compact interval for being a polynomial, the Extreme Value Theorem yields that the minimum and maximum for  $f$  in  $[0, 3]$  exist. Moreover, we also know that these have to be among the critical points in  $(0, 3)$  or the extremes of the interval. By the previous part of the problem, the only critical point that lies in  $(0, 3)$  is  $x = 2$  and  $f(2) = \frac{8}{3} + 2 - 12 + 8 = \frac{2}{3}$ . Moreover,  $f(0) = 8$  and  $f(3) = 9 + \frac{9}{2} - 18 + 8 = 3 + \frac{1}{2}$ .

Hence  $f(2) = \frac{2}{3}$  is the minimum value of  $f$  on  $[0, 3]$  and  $f(0) = 8$  is the maximum value of  $f$  on  $[0, 3]$ .

3. Calculate the integrals

$$\text{a) } \int \frac{2x^3 - 2x^2 + 1}{x-1} dx \qquad \text{b) } \int_0^1 \frac{t \ln(t^2 + 1)}{t^2 + 1} dt$$

**Solution** a) Making the polynomial division one obtains that

$$\frac{2x^3 - 2x^2 + 1}{x-1} = \frac{2x^2(x-1) + 1}{x-1} = 2x^2 + \frac{1}{x-1}.$$

Hence

$$\int \frac{2x^3 - 2x^2 + 1}{x-1} dx = \int 2x^2 dx + \int \frac{1}{x-1} dx = \frac{2}{3}x^3 + \ln|x-1| + C, \quad C \in \mathbb{R}.$$

b) Making the substitution  $y = \ln(t^2 + 1)$ , then  $dy = \frac{2t}{t^2+1} dt$ . Hence

$$\int_0^1 \frac{t \ln(t^2 + 1)}{t^2 + 1} dt = \frac{1}{2} \int_{0=y(0)}^{\ln 2=y(1)} y dy = \frac{\ln^2 2}{4}.$$

4. The expression

$$x^2 \ln y + y^3 e^{-x} = 8,$$

defines  $y$  as a function of  $x$ :  $y = y(x)$ .

(a) Find the value  $y(0)$ .

(b) Find the equation of the tangent line to  $y(x)$  at the point  $P = (0, y(0))$ .

**Solution** (a) Evaluating the expression in  $x = 0$ , we have that

$$y(0)^3 = 8 \Leftrightarrow y(0) = 2.$$

(b) The equation of the tangent line to  $y(x)$  at the point  $P = (0, y(0))$  is given by

$$t(x) = y(0) + y'(0)x = 2 + y'(0)x.$$

Differentiating implicitly on  $x$ , we obtain that

$$2x \ln y + x^2 \frac{y'}{y} + 3y^2 y' e^{-x} - y^3 e^{-x} = 0 \Leftrightarrow y' = \frac{(y^2 e^{-x} - 2x \ln y)y}{x^2 + 3y^4 e^{-x}},$$

which yields

$$y'(0) = \frac{y^4(0)}{3y^3(0)} = \frac{2}{3}.$$

Hence the solution is  $t(x) = 2 + \frac{2x}{3}$ .

5. Let  $F(x, y) = x^4 + y^4 - 36xy$ . Find all stationary points for this function and determine whether they are local maximums, minimums, or saddle points.

**Solution** Observe that the system

$$0 = F_x = 4(x^3 - 9y), \quad 0 = F_y = 4(y^3 - 9x),$$

yields that

$$y = \frac{x^3}{9}.$$

Substituting in the other equation we have that

$$\frac{x^9}{9^3} - 9x = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x^8 = 9^4 = 3^8 \Leftrightarrow x = 0 \quad \text{or} \quad x = \pm 3.$$

Hence, there are three stationary points:  $X_1 = (0, 0)$ ,  $X_2 = (3, 3)$  and  $X_3 = (-3, -3)$ . Observe that

$$F_{xx} = 12x^2, \quad F_{xy} = F_{yx} = -36, \quad F_{yy} = 12y^2.$$

Hence, the determinant of the Hessian is

$$D(x, y) = F_{xx}F_{yy} - F_{xy}^2 = 36(4x^2y^2 - 1).$$

Evaluating on the stationary points we obtain that

$x$	$y$	$F_{xx}$	$D$	Type
0	0	0	$-36 < 0$	Saddle point
3	3	$12 * 9 = 104 > 0$	$36(4 * 81 - 1) > 0$	Local Minimum
-3	-3	$= 104 > 0$	$36(4 * 81 - 1) > 0$	Local Minimum

That is  $X_1$  is a saddle point and  $X_2, X_3$  are local minima.

6. Compute the Taylor polynomial of degree 2 for the function  $g(x) = \sqrt[3]{x+1000}$ , centered at the point  $a = 0$ . Use this polynomial to find an approximate value of  $\sqrt[3]{1003}$ .

**Solution** The solution is given by

$$P(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2$$

Now  $g'(x) = \frac{(x+10^3)^{-\frac{2}{3}}}{3}$  and  $g''(x) = -\frac{2(x+10^3)^{-\frac{5}{3}}}{9}$ . Hence

$$g(0) = 10, \quad g'(0) = \frac{10^{-2}}{3} \quad \text{and} \quad g''(0) = -2\frac{10^{-5}}{9}.$$

Thus  $P(x) = 10 + \frac{10^{-2}}{3}x - \frac{10^{-5}}{9}x^2.$

Now

$$\sqrt[3]{1003} \approx P(3) = 10 + 10^{-2} - 10^{-5} = 10^{-5}(10^6 + 10^3 - 1) = 10^{-5}(1001000 - 1) = 10,00999.$$

NB: The real value is approximatedly **10.009991663..**

7. Solve the following system of linear equations using Cramer's Rule:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 2 \\ -x_2 - 3x_3 &= 1 \\ 2x_1 + 2x_2 + 6x_3 &= 2 \end{aligned}$$

**Solution** Cramer's Rule gives the unique solution

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\mathbf{A}|} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|\mathbf{A}|} \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|\mathbf{A}|}$$

provided that  $|\mathbf{A}| \neq 0$ .

So we compute:

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{vmatrix} = 1 \begin{vmatrix} -1 & -3 \\ 2 & 6 \end{vmatrix} - 2 \begin{vmatrix} 0 & -3 \\ 2 & 6 \end{vmatrix} + 4 \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} \\ &= 1(-6+6) - 2(0 - (-6)) + 4(0+2) = -12 + 8 = -4 \end{aligned}$$

Similar determinant calculations for the numerators yield

$$\begin{aligned} \begin{vmatrix} 2 & 2 & 4 \\ 1 & -1 & -3 \\ 2 & 2 & 6 \end{vmatrix} &= -8 \\ \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & -3 \\ 2 & 2 & 6 \end{vmatrix} &= -8, \\ \begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 2 & 2 \end{vmatrix} &= 4, \end{aligned}$$

so

$$(x_1, x_2, x_3) = \left( \frac{-8}{-4}, \frac{-8}{-4}, \frac{4}{-4} \right) = (2, 2, -1)$$

is the solution.

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