This is Problem set 3, part of the examination of the course. The problems are to be solved individually and are due by April 14th. Please write your solutions clearly and justify every step.

Exercise 1  (6 points)

The aim of this exercise is to prove that if \( f(x) \) is a primitive recursive function and \( R(x,y) \) is a primitive recursive relation (where \( x, y \) are given tuples), then the following relations are primitive recursive:

\[
S(x,y) = \exists x (x \leq f(x) \land R(x,y))
\]

\[
T(x,y) = \forall x (x \leq f(x) \rightarrow R(x,y))
\]

and the following function is also primitive recursive:

\[
q(x,y) = \mu x (x \leq f(x) \land R(x,y))
\]

where \( \mu x \phi(x) \) is the minimum \( x \) such that \( \phi(x) \) holds if there is such number, and 0 otherwise.

1. Define the functions \( ne(x), di(x,y) \) and \( id(x,y) \) as follows:

   (a) \( ne(0) = 1 \) and \( ne(x) = 0 \) for \( x \neq 0 \)
   (b) \( di(0,x) = di(x,0) = 0 \) and \( di(x,y) = 1 \) if \( x \neq 0 \) and \( y \neq 0 \)
   (c) \( id(x,y) = 0 \) if \( x = y \) and \( id(x,y) = 1 \) if \( x \neq y \)

Prove that each one of them is primitive recursive.
2. Define \( j(x, y) \) as follows:

\[
\begin{align*}
  j(0, y) &= 0 \\
  j(n+1, y) &= (n+1)a + j(n, y)ne(a)
\end{align*}
\]

where \( a = ne(ne(r(0, y)))ne(r(n+1, y))ne(j(n, y)) \) and \( r \) is a primitive recursive function such that \( R(x, y) \iff r(x, y) = 0 \) (that exists by hypothesis). Prove that:

\[
  j(n+1, y) = n + 1 \iff \neg R(0, y) \wedge R(n+1, y) \wedge j(n, y) = 0
\]

and that otherwise \( j(n+1, y) = j(n, y) \).

3. Prove that:

\[
q(x, y) = j(f(x), y)
\]

and

\[
S(x, y) \iff R(q((x, y)), y)
\]

and therefore, that \( q \) is a primitive recursive function and \( S \) is a primitive recursive relation.

4. Using the previous item show that \( T(x, y) \) is a primitive recursive relation.

**Exercise 2  (4 points)**

Let \( FR(x) \) denote the primitive recursive relation that says that \( x \) is the Gödel code of a sequence of (Gödel codes of) formulas each of which is atomic or has been obtained from previous elements in the sequence through the operations of disjunction, negation or generalization.

1. Let:

\[
Form(x) \iff \exists n(n \leq (Pr(l(x))^2)^2l(x)^2 \wedge FR(n) \wedge x = l(n)GLn)
\]

where \( Pr(n) \) is the \( n \)-th prime number, \( l(x) \) is the length of \( x \) (i.e., the number of members of the sequence coded by \( x \)) and \( nGLx \) is the \( n \)-th number in the...
sequence coded by \( x \). Using that all these functions are primitive recursive, prove that \( \text{Form}(x) \) is a primitive recursive relation and explain why, if the bound on \( n \) is correct, \( \text{Form}(x) \) if and only if \( x \) is the Gödel code of a formula.

2. Justify the bound \( \text{Pr}(l(x)^2)x^2(x^2) \) as follows: first, prove that the length of the shortest sequence of formulas corresponding to \( x \) can be at most the number of subformulas of \( x \) and that this number is at most \( l(x)^2 \); second, prove that in the prime factorization of \( n \) there are at most \( l(x)^2 \) prime factors, each of which is at most \( \text{Pr}(l(x)^2) \) and whose exponents (which are subformulas of \( x \)) are at most \( x \).

**Exercise 3  (4 points)**

Give an example of a consistent theory that is \( \omega \)-inconsistent. Can it be primitive recursively axiomatized?

**Exercise 4  (6 points)**

Let \( \mathcal{T} \) be a primitive recursively axiomatized consistent extension of \( PA \). Explain under what conditions each of the following statements is true:

1. If \( \mathcal{T} \vdash \phi \), then \( \mathcal{T} \vdash \text{Prov}_\mathcal{T}([\phi]) \)
2. \( \mathcal{T} \vdash \phi \rightarrow \text{Prov}_\mathcal{T}([\phi]) \)
3. If \( \mathcal{T} \vdash \text{Prov}_\mathcal{T}([\phi]) \), then \( \mathcal{T} \vdash \phi \)
4. \( \mathcal{T} \vdash \text{Prov}_\mathcal{T}([\phi]) \rightarrow \phi \)