One-Parameter Bifurcation of equilibria in Two-Dimensional Continuous-time Dynamical Systems

Consider a continuous-time system depending on parameters

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m \]

where \( f \) is smooth with respect to both \( \alpha \) and \( x \). Let \( x = x_0 \) be a hyperbolic equilibrium (i.e. there is no eigenvalues of the Jacobian matrix evaluated at \( x_0 \) on the imaginary axis) in the system for \( \alpha = \alpha_0 \). As we have seen, under a small parameter perturbation the equilibrium moves slightly but remains hyperbolic. Therefore we can vary \( \alpha \) further. Generically there are only two ways in which the hyperbolicity condition can be violated, a simple real eigenvalue approaches zero and we have \( \lambda_1 = 0 \) or a pair of simple complex eigenvalues reaches the imaginary axis and we have \( \lambda_{1,2} = \pm i\omega_0, \omega_0 > 0 \) for some value of the parameter. It is obvious (and can be rigorously formalized) that we need more parameters to allocate extra eigenvalues on the imaginary axis.

In this note we discuss essentially a non-hyperbolic equilibrium for two-dimensional systems with a pair of imaginary eigenvalues. We shall analyze the corresponding bifurcations of local phase portrait under variation of the parameter.

1 Bifurcations and bifurcation diagrams

To be able to give a definition of bifurcations we have to define some equivalence of systems.

**Definition 1.1.** A dynamical system \( \{ T, \mathbb{R}^n, \varphi^t \} \) is called topologically equivalent to a dynamical system \( \{ T, \mathbb{R}^n, \psi^t \} \) if there is continuous map \( h : \mathbb{R}^n \to \mathbb{R}^n \) with continuous inverse mapping orbits of the first dynamical system onto orbits of the second system, preserving the direction of time.

**Definition 1.2.** The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.

So a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

**Example 1** (Saddle-node bifurcation). Consider the one-dimensional system

\[ \dot{x} = \alpha + x^2 =: f(x, \alpha) \quad (1.1) \]

At \( \alpha = 0 \) this system has a nonhyperbolic equilibrium \( x_0 = 0 \) with \( \lambda = f_x(0, 0) = 0 \). The behavior of the system for all other values of \( \alpha \) is also clear.

- For \( \alpha < 0 \) there are two equilibria in the system: \( x_{1,2} = \pm \sqrt{-\alpha} \), the left one of which is stable, while the right one is unstable.
- For \( \alpha > 0 \) there are no equilibria in the system.
• While $\alpha$ crosses zero from negative to positive values, the two equilibria (stable and unstable) collide, forming at $\alpha = 0$ an equilibrium with $\lambda = 0$, and disappear. This is a fold bifurcation.

The term "collision" is appropriate since the speed of approach ($\frac{dx}{d\alpha} x_{1,2}(\alpha)$) of the equilibria tends to $\infty$ as $\alpha \to 0$.

A conventional way to represent this situation graphically is to plot the $x$-positions of the equilibria as a function of the parameter $\alpha$, with the stable equilibrium solid and the unstable equilibrium dashed in the figure.

**Example 2 (Andronov-Hopf bifurcation).** Consider the following system

\[
\begin{align*}
\dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2)
\end{align*}
\] (1.2)

In polar coordinates $(\rho, \theta)$ it takes the form

\[
\begin{align*}
\dot{\rho} &= \rho(\alpha - \rho^2) \\
\dot{\theta} &= 1
\end{align*}
\] (1.3)

and can be integrated explicitly. Since the equations for $\rho$ and $\theta$ are independent, we can easily draw phase portraits of the system in a fixed neighborhood of the origin, a unique equilibrium. Calculations show that

• If $\alpha \leq 0$ the equilibrium is a stable node (sink) since $\dot{\rho} < 0$ and $\rho(t) \to 0$ if we start from any initial point

• If $\alpha > 0$ we have $\dot{\rho} > 0$ for small $\rho > 0$ (the equilibrium becomes an unstable node (source), and $\dot{\rho} < 0$ for sufficiently large $\rho$

It is easy to see from the equation in polar doordinates that the system has a period orbit for any $\alpha > 0$ of radius $\rho_0 = \sqrt{\alpha}$ at $\rho = \rho_0$ we have $\dot{\rho} = 0$. Moreover this periodic orbit is stable since $\dot{\rho} > 0$ inside and $\dot{\rho} < 0$ outside the cycle. Therefore $\alpha = 0$ is a bifurcation parameter value. Indeed, a phase portrait with a limit cycle cannot be deformed by a one-to-one transformation into a phase portrait with only an equilibrium point. The presence of a limit cycle is said to
be a topological invariant. As $\alpha$ increases and crosses zero, we have a bifurcation in the system (4.1) called \textit{Andronov-Hopf bifurcation}. It leads to the appearance, from the equilibrium stable, of small-amplitude periodic oscillations.

As should be clear, an Andronov-Hopf bifurcation can be detected if we fix \textit{any} small neighborhood of the equilibrium. Such bifurcations are called \textit{local}. We will often refer to local bifurcations as \textit{bifurcations of equilibria or fixed points}, although we will analyze not just these points but the whole phase portraits near equilibria. These bifurcations of limit cycles which corresponding to local bifurcations of associated Poincaré maps are called \textit{bifurcations of cycles}. There are also bifurcations that cannot be detected by looking at small neighborhood of equilibrium points or cycles. Such bifurcations are called \textit{global}.

\textbf{Example 3 (Heteroclinic bifurcation).} Consider the following planar system that depends on one parameter:

$$
\begin{align*}
\dot{x}_1 &= 1 - x_1^2 - \alpha x_1 x_2 \\
\dot{x}_2 &= x_1 x_2 + \alpha (1 - x_1^2).
\end{align*}
$$

(1.4)

The system has two saddle equilibria

$$
 x_{(1)} = (-1, 0), \quad x_{(2)} = (1, 0)
$$

for all values of $\alpha$.

- At $\alpha = 0$ the $x_1$-axis is invariant and therefore the saddles are connected by an orbit that is asymptotic to one of them as $t \to +\infty$ and to the other as $t \to -\infty$. Such orbits are called \textit{heteroclinic}. Similarly and orbit that is asymptotic to the same equilibrium as $t \to +\infty$ and $t \to -\infty$ respectively is called \textit{homoclinic}.

- For $\alpha \neq 0$ the $x_1$-axis is no longer invariant, and the connection disappears. This is obviously a global bifurcation. To detect this bifurcation we must fix a region $U$ covering both saddles.

\textbf{Definition 1.3.} The bifurcation associated with the appearance of $\lambda_1 = 0$ is called a \textit{fold} (or \textit{tangent, saddle-node}) bifurcation. The bifurcation corresponding to the presence of $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, is called a \textit{Hopf}, or \textit{(Andronov-Hopf)} bifurcation.
Note that the saddle-node bifurcation is possible if $n \geq 1$, but the Hopf bifurcation needs $n \geq 2$.

There are global bifurcations in which certain local bifurcations are involved. In such cases looking at local bifurcation provides only partial information on the behavior of the system. This possibility is illustrated by the following example.

**Example 4.**

\[
\begin{align*}
\dot{x}_1 &= x_1(1-x_1^2-x_2^2) - x_2(1+\alpha + x_1) \\
\dot{x}_2 &= x_1(1+\alpha + x_1) + x_2(1-x_1^2-x_2^2)
\end{align*}
\]  
(1.5)

where $\alpha$ is a parameter. In polar coordinates $(\rho, \theta)$ system (1.5) takes the form

\[
\begin{align*}
\dot{\rho} &= \rho(1-\rho^2) \\
\dot{\theta} &= 1 + \alpha + \rho \cos \theta
\end{align*}
\]  
(1.6)

Fix a thin annulus $U$ around the unit circle $\{(\rho, \theta) : \rho = 1\}$. At $\alpha = 0$, there is a nonhyperbolic equilibrium point in the annulus:

$$x_0 = (\rho_0, \theta_0) = (1, \pi)$$

It has eigenvalues $\lambda_1 = 0, \lambda_2 = -2$. For small positive values of $\alpha$ the equilibrium disappears, while for small negative values of $\alpha$ it splits into a saddle and a node (this bifurcation is called saddle-node or em fold bifurcation as shown above). This is a local event. However for $\alpha > 0$ a stable limit cycle appears in the system coinciding with the unit circle. This circle is always an invariant set in the system, but for $\alpha \leq 0$ it contains equilibria. Looking at only a small neighborhood of the nonhyperbolic equilibrium we miss the global appearance of the cycle. Notice that at $\alpha = 0$ there is exactly one orbit that is homoclinic to the nonhyperbolic equilibrium $x_0$.

Now we turn to a general discussion of bifurcation in a parameter dependent system. Take some value $\alpha = \alpha_0$ and consider a maximal connected parameter set (called *statum*) counting $\alpha_0$ and composed by those points for which the system has a phase portrait that is topologically equivalent to that at $\alpha = 0$. Taking all such strata in the parameter space $\mathbb{R}^m$, we obtain
the parametric portrait of the system. For example, the system exhibiting the Andronov-Hopf bifurcation has a parameter portrait with two strata: \( \{ \alpha \leq 0 \} \) and \( \{ \alpha > 0 \} \). In the system (1.4) there are three strata: \( \{ \alpha < 0 \} \), \( \{ \alpha = 0 \} \) and \( \{ \alpha > 0 \} \). Note however that the phase portrait of this system for \( \alpha < 0 \) is topologically equivalent to that for \( \alpha > 0 \).

The parametric portrait together with its characteristic phase portraits constitute a bifurcation diagram.

**Definition 1.4.** A bifurcation diagram of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.

**Example 5** (Pitchfork bifurcation). Consider the system

\[
\dot{x} = \alpha x - x^3, \quad x \in \mathbb{R}, \alpha \in \mathbb{R}
\]

The system has an equilibrium \( x_0 = 0 \) for all \( \alpha \). This equilibrium is stable for \( \alpha < 0 \) and unstable for \( \alpha > 0 \) (\( \alpha \) is the eigenvalue of the equilibrium). For \( \alpha > 0 \), there are two extra equilibria branching from the origin (namely \( x_{1,2} = \pm \sqrt{\alpha} \)) which are stable. This bifurcation is often called a pitchfork bifurcation, the reason for which becomes apparent if one has a look at the bifurcation diagram presented in \((x, \alpha)\)-space.

In the simplest cases, the parameter portrait is composed by a finite number of regions in \( \mathbb{R}^m \). Inside each region the phase portraits are topologically equivalent. These regions are separated by bifurcation boundaries, which are "smooth curves, surfaces" in \( \mathbb{R}^m \). The boundaries can intersect, or meet. These intersections subdivide the boundaries into subregions, and so on. A bifurcation boundary is defined by specifying a phase object (equilibrium, cycle, etc) and some bifurcation conditions determining the type of its bifurcation (Hopf, saddle-node, etc). For example the Andronov-Hopf bifurcation of an equilibrium is characterized by one bifurcation condition, the presence of a purely imaginary pair of eigenvalues of the Jacobian matrix evaluated at this equilibrium. When a boundary is crossed, the bifurcation occurs.
Definition 1.5. The codimension of a bifurcation in the system $\dot{x} = f(x, u)$ is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary.

Clearly, it is equivalent to the number of independent conditions determining bifurcation. It makes it clear that the codimension of a certain bifurcation is the same in all generic systems depending on a sufficient number of parameters.

2 Topological normal forms for bifurcations

Definition 2.1. Dynamical systems $\dot{x} = f(x, \alpha)$ ($x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m$) and $\dot{y} = g(y, \beta)$ ($y \in \mathbb{R}^n, \beta \in \mathbb{R}^m$) are topologically equivalent if

1. there is a continuous mapping of parameter space with continuous inverse $p : \mathbb{R}^m \rightarrow \mathbb{R}^m$, $\beta = p(\alpha)$;

2. there is a parameter-dependent continuous mapping with continuous inverse of the phase space $h_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$, ($y = h_\alpha(x)$), mapping orbits of the first system at parameter values $\alpha$ onto the orbits of the second system at parameter values $\beta = p(\alpha)$, preserving the direction of time.
The two systems are called locally topologically equivalent near the origin, if there exists a map \((x, \alpha) \mapsto (h_\alpha(x), p(\alpha))\), defined in a small neighborhood of \((x, \alpha) = (0, 0)\) in the direct product \(\mathbb{R}^n \times \mathbb{R}^m\) and such that

1. \(p : \mathbb{R}^m \to \mathbb{R}^m\) is a continuous mapping with continuous inverse defined in a small neighborhood of \(\alpha = 0\), \(p(0) = 0\);
2. \(h_\alpha : \mathbb{R}^n \to \mathbb{R}^n\) is a parameter-dependent continuous mapping, with the property that the inverse is also continuous, defined in a small neighborhood of \(U_\alpha\) of \(x = 0\), \(h_0(0) = 0\), and mapping orbits of the first system in \(U_\alpha\) onto the orbits of the second system in \(h_\alpha(U_\alpha)\), preserving the direction of time.

We can now consider the problem of classification of all possible bifurcation diagrams of generic systems at least locally and up to and including certain codimension. These local diagrams could then serve as "building blocks" to construct the global bifurcation diagram of any system. Sometimes it is possible to construct a simple (polynomial in \(\xi\)) system

\[
\dot{\xi} = g(\xi, \beta; \sigma), \xi \in \mathbb{R}^n, \beta \in \mathbb{R}^k, \sigma \in \mathbb{R}^l
\]  

(2.1)

which has at \(\beta = 0\) an equilibrium \(\xi = 0\) satisfying \(k\) bifurcation conditions determining a codim \(k\) bifurcation of this equilibrium. Here \(\sigma\) is a vector of the coefficients \(\sigma_i, i = 1, 2, \ldots, l\) of the polynomials involved in (2.1). Normally \(\sigma\) takes integer numbers.

**Definition 2.2** (Topological normal form). System (2.1) is called a topological normal form for the bifurcation if any generic system \(\dot{x} = f(x, \alpha), x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m\) with the equilibrium \(x = 0\) satisfying the same bifurcation conditions at \(\alpha = 0\) is locally topologically equivalent near the origin to (2.1) for some values of the coefficients \(\sigma\).

We have to explain what a generic system means. In all cases we will consider "generic" mens that the system satisfies a finite number of genericity conditions. These conditions will have the form of nonequalities: \(N_i(f) \neq 0, i = 1, 2, \ldots, s\) where each \(N_i\) is some (algebraic) function of certain partial derivatives of \(f\) with respect to \(x\) and \(\alpha\) evaluated at \((x, \alpha) = (0, 0)\). Thus a "typical" parameter-dependent system satisfies these conditions. Actually the values of \(\sigma\) is then determined by values of \(N_i, i = 1, 2, \ldots, s\).

It is useful to distinguish those genericity conditions which are determined by the system at critical parameter values \(\alpha = 0\).

- **Nondegeneracy conditions**: conditions can be expressed in terms of partial derivatives of \(f(x, 0)\) w.r.t. \(x\) evaluated at \(x = 0\).
- **Transversality conditions**: conditions in which the derivatives of \(f(x, \alpha)\) w.r.t. the parameters \(\alpha\) are involved.

They play different roles. The nondegeneracy conditions guarantee that the critical equilibrium (singularity) is not too degenerate (i.e. typical in a class of equilibria satisfying given bifurcation conditions) while the transversality conditions assure that the parameters "unfold" this singularity in a generic way.
Example 6. The system exhibits Andronov-Hopf bifurcation corresponding to the case $\sigma = -1$ in the two-dimensional topological normal form for this bifurcation:

\[
\begin{align*}
\dot{\xi}_1 &= \beta \xi_1 - \xi_2 + \sigma \xi_1 (\xi_1^2 + \xi_2^2) \\
\dot{\xi}_2 &= \xi_1 + \beta \xi_2 + \sigma \xi_2 (\xi_1^2 + \xi_2^2)
\end{align*}
\]

The transversality condition $\frac{d}{d\alpha} \text{Re} \lambda_{1,2}(\alpha) \big|_{\alpha=0} \neq 0$ means that the pair of complex-conjugate eigenvalues $\lambda_{1,2}(\alpha)$ crosses the imaginary axis with zero speed. The nondegeneracy condition $l_1(0) \neq 0$ implies that a certain combination of Taylor coefficients of the right-hand side of the system (up to including third order coefficients) does not vanish. An explicit form of $l_1(0)$ will be derived later where we will also prove that the above system is really a topological normal form for the Hopf bifurcation. We will also prove that $\sigma = \text{sign} l_1(0)$.

3 The saddle-node bifurcation

3.1 The normal form of the saddle-node bifurcation

Consider the following one-dimensional dynamical system depending on one parameter:

\[
\dot{x} = \alpha + x^2 \equiv f(x, \alpha)
\]

We know already the diagram curve and bifurcation of the system from Example 1. We can claim that the system $\dot{x} = \alpha - x^2$ can be considered in the same way as in the Example 1. The analysis reveals two equilibria for $\alpha > 0$.

Now we are going to study adding to the system $\dot{x} = \alpha + x^2 \equiv f(x, \alpha)$ higher order terms

\[
\dot{x} = \alpha + x^2 + O(x^3)
\]

that can depend smoothly on the parameter. It happens that these terms do not change qualitatively the behavior of the system near the origin $x = 0$ for parameter values close to $\alpha = 0$.

First we do an analysis on equilibria. Introduce a scalar variable $y$ and write the equation (3.1) as

\[
\dot{y} = F(y, \alpha) = \alpha + y^2 + \psi(y, \alpha)
\]

where $\psi = O(y^3)$ is a smooth function of $(y, \alpha)$ near $(0, 0)$. Consider the equilibrium manifold of the above equation near the origin $0$ of the $(y - \alpha)$-plane:

\[
M = \{(y, \alpha) : F(y, \alpha) = \alpha + y^2 + \psi(y, \alpha) = 0\}
\]

The curve $M$ passes through the origin $F(0, 0) = 0$. By the Implicit Function Theorem (since $F_\alpha(0, 0) = 1$), it can be locally parametrized by $y$:

\[
M = \{(y, \alpha) : \alpha = g(y)\}
\]

where $g$ is smooth and defined for small $|y|$. Moreover,

\[
g(y) = -y^2 + O(y^3)
\]
Thus for any sufficientlt small $\alpha < 0$ there are two equilibria of (3.2) near the origin, $y_1(\alpha)$ and $y(\alpha)$, which are close to equilibria of the original system $x_1(\alpha) = \sqrt{-\alpha}$ and $x_2(\alpha) = -\sqrt{-\alpha}$ for same parameter value.

Next we construct a continuous map whose inverse is also continuous. For small $|\alpha|$, construct a parameter-dependent map $y = h_\alpha(x)$ as follows.

$$h_\alpha(x) = \begin{cases} x & \text{for } \alpha \geq 0 \\ a(\alpha) + b(\alpha)x & \text{for } \alpha < 0 \end{cases}$$

where $a, b$ ar uniquely determined by the conditions

$$h_\alpha(x_j(\alpha)) = y_j(\alpha), \quad j = 1, 2$$

The constructed map $h_\alpha : \mathbb{R} \to \mathbb{R}$ is continous and invertibel and its inverse is also continuous. It maps orbits of the system $\dot{x} = \alpha + x^2$ near the origin into the corresponding orbits of (3.2), preserving the direction of time. So this is a topological equivalence. Hence we have prove the following lemma.

**Lemma 3.1.** The system $\dot{x} = \alpha + x^2 + O(x^3)$ is locally topologically equivalent near the origin to the system $\dot{x} = \alpha + x^2$.

### 3.2 Generic Saddle-node bifurcation

Suppose that the system

$$\dot{x} = f(x, \alpha), x \in \mathbb{R}, \alpha \in \mathbb{R}$$

with smooth $f$. It has at $\alpha = 0$ the equilibrium $x = 0$ with $\lambda = f_x(0, 0) = 0$. Expand $f(x, \alpha)$ as a Taylor series with respect to $x$ at $x = 0$:

$$f(x, \alpha) = f_0(\alpha) + f_1(\alpha)x + f_2(\alpha)x^2 + O(x^3)$$

Two conditions are satisfied: $f_0(0) = f(0, 0) = 0$ (equilibrium condition) and $f_1(0) = f_x(0, 0) = 0$ (saddle-node bifurcation condition).
By suitable variable and parameter changes we want to arrive to a simpler system whose behavior is easy to analyze.

First we make a shift of the coordinate: \( \xi = x + \delta \) where \( \delta \) is to be determined. Substituting the into the system yields

\[
\dot{\xi} = \dot{x} = f_0(\alpha) + f_1(\alpha)(\xi - \delta) + f_2(\alpha)(\xi - \delta)^2 + \cdots
\]

Therefore,

\[
\dot{\xi} = [f_0(\alpha) - f_1(\alpha)\delta + f_2(\alpha)\delta^2 + O(\delta^3)] + [f_1(\alpha) - 2f_2(\alpha)\delta + O(\delta^2)]\xi
\]

\[+ [f_2(\alpha) + (\delta)^3]{\xi^2} + O(\delta^3)\]

Assume

\[
f_2(0) = \frac{1}{2} f_{xx}(0,0) \neq 0
\]

Then there is a smooth function \( \delta(\alpha) \) that annihilates the linear term in the above equation for all sufficiently small \( |\alpha| \). This can be justified with the Implicit Function Theorem. In fact, the condition for linear term to vanish can be written as

\[
F(\alpha, \delta) \equiv f_1(\alpha) - 2f_2(\alpha)\delta + \delta^2\psi(\alpha, \delta) = 0
\]

with some smooth function \( \psi \). We have

\[
F(0,0) = 0, \quad \frac{\partial F}{\partial \delta}(0,0) = -2f_2(0) \neq 0, \quad \frac{\partial F}{\partial \alpha}(0,0) = f'_1(0),
\]

which implies (local) existence and uniqueness of a smooth function \( \delta = \delta(\alpha) \) such that \( \delta(0) = 0 \) and \( F(\alpha, \delta(\alpha)) \equiv 0 \). It also follows that

\[
\delta(\alpha) = \frac{f'_1(0)}{2f_2(0)} \alpha + O(\alpha^2)
\]

The equation for \( \xi \) now contains no linear terms:

\[
\dot{\xi} = [f_0(0)\alpha + O(\alpha^2)] + [f_2(0) + O(\alpha)]\xi^2 + O(\xi^3)
\]

Next introduce a new parameter. Consider as a new parameter \( \mu = \mu(\alpha) \) the constant (\( \xi \)-independent) term in the previous equation:

\[
\mu = f''_0(0)\alpha + \alpha^2\phi(\alpha)
\]

where \( \phi \) is smooth. We have

(a) \( \mu(0) = 0 \)

(b) \( \mu'(0) = f'_0(0) = f_\alpha(0,0) \).

If we assume that \( f_\alpha(0,0) \neq 0 \) then the Inverse Function Theorem implies local existence and uniqueness of a smooth inverse function \( \alpha = \alpha(\mu) \) with \( \alpha(0) = 0 \). Therefore \( \xi \)-equation becomes

\[
\dot{\xi} = \mu + b(\mu)\xi^2 + O(\xi^3)
\]
where $b(\mu)$ is a smooth function with $b(0) = f_2(0) \neq 0$ due to the assumption $f_2(0) \neq 0$.

At last we rescale the variable $\eta = |b(\mu)|\xi$ and $\beta = |b(\mu)|\mu$. Then we get

$$\dot{\eta} = \beta + sn^2 + O(\eta^3)$$

where $s = \text{sign}B(0) = \pm 1$. Hence we have proved the following theorem.

**Theorem 3.2.** Suppose that a one-dimensional system

$$\dot{x} = f(x, \alpha), x \in \mathbb{R}, \alpha \in \mathbb{R}$$

with smooth $f$, has at $\alpha = 0$ the equilibrium $x = (0, 0)$, and let $\lambda = f_x(0, 0) = 0$. Assume that the following conditions are satisfied:

(SN.1) $f_{xx}(0, 0) \neq 0$;
(SN.2) $f_{\alpha}(0, 0) \neq 0$.

Then there are invertible coordinate and parameter changes transforming the system into

$$\dot{\eta} = \beta \pm \eta^2 + O(\eta^3)$$

Using the previous lemma we can eliminate $O(\eta^3)$ terms and finally arrive at the following general result.

**Theorem 3.3** (Topological normal form for saddle-node bifurcation). Any generic scalar one-parameter system $\dot{x} = f(x, \alpha)$ having at $\alpha = 0$ the equilibrium $x = 0$ with $\lambda = f_x(0, 0) = 0$, is locally topologically equivalent near the origin to one of the following normal forms:

$$\dot{\eta} = \beta \pm \eta^2.$$

4 The Andronov-Hopf bifurcation

4.1 The normal form of the Hopf bifurcation

Consider (4.1) depending on $\alpha$. This system has the equilibrium $x_1 = x_2 = 0$ for all $\alpha$ with the Jacobian matrix

$$A = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}$$

having eigenvalues $\lambda_{1,2} = \alpha \pm i$. Introduce the complex variable $z = x_1 + ix_2$, $\bar{z}z = x_1 - ix_2$, $|z|^2 = z\bar{z} = x_1^2 + x_2^2$. This variable satisfies the differential equation

$$\dot{z} = (\alpha + i)z - z|z|^2$$

Finally using the representation $z = re^{i\varphi}$ we obtain the decoupled system in polar coordinates (1.3)

We have already seen the dynamical behavior under variation of $\alpha$. Note that at $\alpha = 0$ the equilibrium is nonlinearly stable and topologically equivalent to the focus, which is also called a weakly attracting focus. This equilibrium is surrounded for $\alpha > 0$ by an isolated closed orbit (limit cycle) that is unique and stable.
This bifurcation can also be presented in $x, y, \alpha$-space. The appearing $\alpha$-family of limit cycles form a paraboloid surface.

A system having nonlinear terms with the opposite sign,

\[
\begin{align*}
\dot{x}_1 &= \alpha x_1 - x_2 + x_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= x_1 + \alpha x_2 + x_2(x_1^2 + x_2^2)
\end{align*}
\]

which has the complex form

\[
\dot{z} = (\alpha + i)z + |z|^2
\]

can be analyzed in the same way. The system undergoes a Hopf bifurcation at $\alpha = 0$. Contrary to the system with negative sign in the nonlinear term there is an unstable limit cycle in this system. It disappears when $\alpha$ crosses zero from negative to positive values. The equilibrium at the origin has the same stability for $\alpha \neq 0$ as in (4.1): It is stable for $\alpha < 0$ and unstable for $\alpha > 0$. Its stability at the critical parameter value is opposite to that in (4.1): It is nonlinearly unstable at $\alpha = 0$.

![Diagram showing subcritical Hopf bifurcations](image)

We have seen two types of Andronov-Hopf bifurcations. The first is called supercritical and the second is called subcritical. The following figures illustrate clearly the Andronov-Hopf bifurcation under variation of $\alpha$. The left is supercritical and the right is subcritical.

Let us now add some higher order terms on the system (4.1) and write it in matrix form

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
\alpha & -1 \\
1 & \alpha
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} \left( x_1^2 + x_2^2 \right) + O(\|x\|^4),
\]

where $x = (x_1, x_2)^T$, $\|x\|^2 = x_1^2 + x_2^2$ and $O(\|x\|^4)$ terms depends smoothly on $\alpha$. Then we can prove the following lemma.

**Lemma 4.1.** Suppose (4.2) is locally topologically equivalent near the origin to system (4.1).

### 4.2 Generic Andronov Hopf bifurcation

Now we consider generic two-dimensional system undergoing an Andronov-Hopf bifurcation. We assume that $f$ is smooth and has at $\alpha = 0$ the equilibrium $x = 0$ with eigenvalues $\lambda_{1,2} = \pm i\omega_0$. 
\( \omega_0 > 0 \). By the implicit function theorem the system has a unique equilibrium \( x_0(\alpha) \) in some neighborhood of the origin for sufficiently small \( |\alpha| \) since \( \lambda = 0 \) is not an eigenvalue of the Jacobian matrix. Perform a coordinate change we can shift the equilibrium to origin. So we assume that \( x = 0 \) is the equilibrium of the system for sufficiently small \( |\alpha| \). Thus it can be written as

\[
\dot{x} = A(\alpha)x + F(x, \alpha)
\]

where \( F = (F_1, F_2)^T \) is a smooth vector-valued function whose components \( F_{1,2} \) have Taylor expansion in \( x \) starting with at least quadratic terms, \( F = O(||x||^2) \). The Jacobian matrix \( A(\alpha) \) can be written as

\[
A(\alpha) = \begin{pmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{pmatrix}
\]

with smooth functions of \( \alpha \) as its elements. Its eigenvalues are the root of the characteristic equation

\[
\lambda^2 - \sigma \lambda + \Delta = 0
\]

where \( \sigma = \sigma(\alpha) = a(\alpha) + d(\alpha) = \text{tr}A(\alpha) \), and \( \Delta = \Delta(\alpha) = a(\alpha)d(\alpha) - b(\alpha)c(\alpha) = \text{det}A(\alpha) \). The eigenvalues are

\[
\lambda_{1,2} = \frac{1}{2}(\sigma(\alpha) \pm \sqrt{\sigma^2(\alpha) - 4\Delta(\alpha)})
\]

The Andronov-Hopf bifurcation condition implies that

\[
\sigma(0) = 0, \Delta(0) = \omega_0^2 > 0.
\]

For small \( |\alpha| \) we can introduce

\[
\mu(\alpha) = \frac{\sigma(\alpha)}{2}, \omega(\alpha) = \frac{1}{2} \sqrt{4\Delta(\alpha) - \sigma^2(\alpha)}
\]
and therefore obtain the following representation for the eigenvalues:

\[ \lambda_1(\alpha) = \lambda(\alpha), \lambda_2(\alpha) = \overline{\lambda(\alpha)}, \]

where

\[ \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0. \]

Now let \( q(\alpha) \in \mathbb{C}^2 \) be an eigenvector of \( A(\alpha) \) corresponding to the eigenvalue \( \lambda(\alpha) \):

\[ A(\alpha)q(\alpha) = \lambda(\alpha)q(\alpha), \]

and let \( p(\alpha) \in \mathbb{C}^2 \) be an eigenvector of the transposed matrix \( A(\alpha) \) corresponding to its eigenvalue \( \overline{\lambda(\alpha)} \):

\[ A^T(\alpha)p(\alpha) = \overline{\lambda(\alpha)}p(\alpha). \]

It is always possible to normalize the vector \( p(\alpha) \) with respect to \( q(\alpha) \): \( \langle p(\alpha), q(\alpha) \rangle = 1 \), where \( \langle \cdot, \cdot \rangle = \bar{p}_1q_1 + \bar{p}_2q_2 \). Now \( p \) and \( q \) are linear independent in \( \mathbb{C}^2 \), we have for any \( x \in \mathbb{R}^2 \)

\[ x = zq(\alpha) + \bar{z}\bar{q}(\alpha) \]

for some complex number \( z \), provided the eigenvectors are specified. Take scalar product between \( p \) and the both side of this equation we obtain,

\[ \langle p(\alpha), x \rangle = \langle p(\alpha), zq(\alpha) \rangle + \langle p(\alpha), \bar{z}\bar{q}(\alpha) \rangle = z\langle p(\alpha), q(\alpha) \rangle + \overline{z}\langle p(\alpha), \bar{q}(\alpha) \rangle. \]

Since \( p \) and \( \bar{q} \) are eigenvectors corresponding to different eigenvectors \( \langle p(\alpha), \bar{q}(\alpha) \rangle = 0 \), and hence we have an explicit formula to determine \( z \):

\[ z = \langle p(\alpha), x \rangle. \]

Now differentiating \( z \):

\[ \dot{z} = \langle p(\alpha), \dot{x} \rangle = \langle p(\alpha), A(\alpha)x + F(x, \alpha) \rangle \\
= \langle p(\alpha), A(\alpha)(zq(\alpha) + \bar{z}\bar{q}(\alpha)) + F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle \\
= \langle p(\alpha), z\lambda(\alpha)q(\alpha) + \bar{z}\lambda(\alpha)\bar{q}(\alpha) + F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle \\
= z\lambda(\alpha) + \langle p(\alpha), F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle. \]

Let now

\[ g(z, \bar{z}, \alpha) = \langle p(\alpha), F(zq(\alpha) + \bar{z}\bar{q}(\alpha), \alpha) \rangle = O(|z|^2) \]

Therefore we have proved

**Lemma 4.2.** By introducing a complex variable \( z = \langle p(\alpha), x \rangle \) the system \( \dot{x} = A(\alpha)x + F(x, \alpha) \) can be rewritten for sufficiently small \( |\alpha| \) as a single equation

\[ \dot{z} = \lambda(\alpha)z + g(z, \bar{z}, \alpha) \]

where \( g(z, \bar{z}, \alpha) = O(|z|^2) \) is smooth (of \( z, \bar{z}, \alpha \)).
Next we write \( g \) as a formal Taylor series in two complex variables \( z \) and \( \bar{z} \):

\[
g(z, \bar{z}, \alpha) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(\alpha) z^k \bar{z}^l
\]

where

\[
g_{kl}(\alpha) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} (p(\alpha), F(zq(\alpha) + \bar{z}q(\alpha), \alpha)|_{z=0}, k + l \geq 2, k, l = 0, 1, 2, \ldots
\]

Note that the representation \( x = zq(\alpha) + \bar{z}q(\alpha) \) is a linear relation between \( x_1, x_2 \) and the real and imaginary parts of \( z \). Thus the introduction of \( z \) can be viewed as a linear invertible change of variables \( y = T(\alpha)x \) and taking \( z = y_1 + iy_2 \). As in the real eigenbasis of \( A(\alpha) \) composed by \( \{2\Re q, -2\Im q\} \). In this basis the matrix \( A(\alpha) \) has its canonical real (Jordan) form:

\[
J(\alpha) = T(\alpha)A(\alpha)T^{-1}(\alpha) = \begin{pmatrix}
\mu(\alpha) & -\omega(\alpha) \\
\omega(\alpha) & \mu(\alpha)
\end{pmatrix}
\]

Note also that the coefficients \( g_{kl} \) can be computed by higher order terms of the original system, i.e. \( F(x, 0) \). Assume it has the representation

\[
F(x, 0) = \frac{1}{2} B(x, x) + \frac{1}{6} C(x, x, x) + O(\|x\|^4)
\]

where \( B(x, y) \) and \( C(x, y, u) \) are symmetric multilinear vector-valued functions of \( x, y, u \in \mathbb{R}^2 \). More precisely,

\[
B(x, y) = \sum_{j,k=1}^{2} \frac{\partial^2 F(\xi, 0)}{\partial \xi_j \partial \xi_k} |_{\xi=0} x_j y_k, \quad i = 1, 2.
\]

and

\[
C(x, y, u) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F(\xi, 0)}{\partial \xi_j \partial \xi_k \partial \xi_l} |_{\xi=0} x_j y_k u_l, \quad i = 1, 2.
\]

Then

\[
B(zq + \bar{z}q, zq + \bar{z}q) = z^2 B(q, q) + 2z\bar{z} B(q, \bar{q}) + \bar{z}^2 B(\bar{q}, \bar{q}),
\]

where \( q = q(0), p = p(0) \), so the Taylor coefficients \( g_{kl}, k + l = 2 \) of the quadratic terms in \( g(z, \bar{z}, 0) \) can be expressed by

\[
g_{20} = \langle p, B(q, q) \rangle, \quad g_{11} = \langle p, B(q, \bar{q}) \rangle, \quad g_{02} = \langle p, B(\bar{q}, \bar{q}) \rangle,
\]

and similar calculations for \( C \):

\[
g_{21} = \langle p, C(q, q, \bar{q}) \rangle.
\]

After a lengthy computation and argument we can show the following two theorems.

**Theorem 4.3.** Assume that a two dimensional system \( \frac{dx}{dt} = f(x, \alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R} \) with smooth \( f \), has for all sufficiently small \( |\alpha| \) the equilibrium \( x = 0 \) with eigenvalues \( \lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha) \), where \( \mu(0) = 0, \omega(0) = \omega_0 > 0 \). Let the following conditions hold:
(AH.1) (nondegeneracy condition) \( l_1(0) \neq 0 \),
where \( l_1(\alpha) \) is the first Lyapunov coefficient and
\[
l_1(0) = \frac{1}{2\omega_0} \text{Re}(ig_{20}g_{11} + \omega_0 g_{21})
\]
(AH.2) (transversality condition) \( \mu'(0) \neq 0 \).
Then there are invertible coordinate and parameter changes and a time reparametrization trans-
forming the system into
\[
\frac{d}{d\tau} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + s(y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\|y\|^4)
\]
where \( s = \text{sign} l_1(0) = \pm 1 \).

Note that if we choose \( q = p = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \) then
\[
l_1(0) = \frac{1}{8\omega_0} (P_{uuu} + P_{uvv} + Q_{uuu} + Q_{vvv}) + \frac{1}{\omega_0} \left[ P_{uu}(P_{uu} + P_{vv}) - Q_{uu}(Q_{uu} + Q_{vv}) - P_{uu}Q_{uu} + P_{vv}Q_{vv} \right]
\]
and the lower indices stand for partial derivatives evaluated at \( x = 0 \), and \( P \ Q \) satisfy
\[
F(x, 0) = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} P(u, v) \\ Q(u, v) \end{pmatrix}.
\]
Note also that different choice of \( q \) will lead to different \( l_1 \) however the sign is same for all choice of \( q \), which is the most important thing to remember.

The condition (AH.1) assures that the transformation for coordinate changes and (AH.2) prove the coefficient to the cubic term to become zero.

**Theorem 4.4** (Topological normal form for Andronov-Hopf bifurcation). Any generic two-
dimensional, one-parameter system \( \dot{x} = f(x, \alpha) \) (i.e. satisfying (AH.1) and (AH.2)) having at \( \alpha = 0 \) the equilibrium \( x = 0 \) with eigenvalues
\[
\lambda_{1,2}(0) = \pm i\omega_0, \omega_0 > 0
\]
is locally topologically equivalent near the origin to one of the following normal forms:
\[
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + s(y_1^2 + y_2^2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\]
where \( s = \text{sign} \ l_1(0) = \pm 1 \).

**Example 7** (Andronov-Hopf bifurcation in a predator-pray model). This is a simple model of
an ecological system with three components: a plant, a small mammal called the Murat which
eats the plant, and a carnivorous predator called the Vekton which eats the mammal.
We are using the model to test our belief that we can control this ecosystem by controlling the
plant population. Our goal is to maintain both species at healthy numbers. We denote the
The population of the plant eater by $M$ and the population of the carnivore by $V$. The basic equations governing this system can be set up as follows:

\[
\begin{align*}
\frac{dM}{dt} &= rM \left(1 - \frac{M}{A}\right) - \frac{cMV}{M + H}, \\
\frac{dV}{dt} &= \frac{cMV}{M + H} - dV
\end{align*}
\]

The parameters were explained in class, so we just review them very briefly here. The parameter $A$ is the maximum sustainable population of the plant-eaters in the absence of the carnivores, and the model incorporates this via a logistic law. The parameter $A$ is the one we control, by controlling the plant population, and this is our only management tool for this system. In the limit $M \ll A$, the constraint represented by $A$ is unimportant, and the net birth rate of the plant eaters is $r$, and they will grow like $e^{rt}$ until $M$ becomes comparable with $A$. The second, negative, term in the $M$-equation models the effects of the predators. The loss is proportional to the product of the populations, because this product is proportional to the encounter rate. The denominator is a saturation effect, accounting for the fact that if the population of plant eaters is large, predators won’t be as hungry and there will be fewer kills per encounter. In the predator equation, the first term models the birth rate, and accounts for the fact that it will be higher if food is more plentiful. If food is very plentiful ($M$ much larger than $H$), this birth term saturates with a rate $dV$, so $d$ is the natural birth rate with ample food. It is an artificial simplification of the model that the saturation parameter has the same value $H$ in both equations.

Now we simplify the system a little further. Let $x_1 = M/A$, and $x_2 = V/A$ and choose $\alpha = H/A$ as control parameter. So the equations are

\[
\begin{align*}
\dot{x}_1 &= rx_1 (1 - x_1) - \frac{cx_1 x_2}{x_1 + \alpha}, \\
\dot{x}_2 &= -dx_2 + \frac{cx_1 x_2}{x_1 + \alpha}
\end{align*}
\]

We assume that $c > d$. If we consider $x_1 > -\alpha$ then the system will be orbitally equivalent \(^1\) to a polynomial system. So a time scaling by $\alpha + x_1$, i.e. $dt = (x_1 + \alpha)d\tau$. Thus we shall study the following polynomial system.

\[
\begin{align*}
\dot{x}_1 &= rx_1 (1 - x_1)(\alpha + x_1) - cx_1 x_2, \\
\dot{x}_2 &= -\alpha dx_2(c - d)x_1 x_2
\end{align*}
\]

This system has a nontrivial equilibrium

\[
E_0 = \left(\frac{\alpha d}{c - d}, -\frac{r\alpha}{c - d} \left(1 - \frac{\alpha d}{c - d}\right)\right).
\]

The Jacobian evaluated at this point is

\[
A(\alpha) = \begin{pmatrix}
\frac{\alpha rd(c + d)}{(c - d)^2} & \frac{c - d}{c + d} - \alpha \\
\frac{\alpha rd(c + d)}{c - d} & 0
\end{pmatrix} - \frac{\alpha cd}{c - d}.
\]

\(^1\)Two systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ satisfying $f(x) = \mu(x)g(x)$ for all $x \in \mathbb{R}^n$ for a smooth positive function $\mu$ are called orbitally equivalent.
and thus
\[ \mu(\alpha) = \sigma(\alpha) = \frac{\alpha rd(c + d)}{2(c - d)^2} \left( \frac{c - d}{c + d} - \alpha \right) \]

We have \( \mu(\alpha_0) = 0 \) for
\[ \alpha_0 = \frac{c - d}{c + d} \]

Moreover,
\[ \omega^2(\alpha_0) = \frac{rd(c - d)}{(c + d)^3} > 0 \]

Thus at \( \alpha = \alpha_0 \) the equilibrium \( E_0 \) has eigenvalues \( \lambda_{1,2}(\alpha_0) = \pm i\omega(\alpha_0) \) and an Andronov-Hopf bifurcation takes place. (Note that we cannot use this \( \omega_0 \) to direct determine the period around \( E_0 \) in the original system since we study an orbitally equivalent system.) The equilibrium is

- stable if \( \alpha > \alpha_0 \)
- unstable if \( \alpha < \alpha_0 \)

Note that the critical value of \( \alpha \) corresponds to the passing of the line defined by \( \dot{x}_2 = 0 \) through the maximum of the curve defined by \( \dot{x}_1 = 0 \). Thus if the line \( \dot{x}_2 = 0 \) is to the right of the maximum the point is stable, while if this line is to the left, the point is unstable.

To apply the normal form theorem to the analysis of this Hopf bifurcation, we have to check the generacy conditions. The transversality condition (AH.1) is easy to check:
\[ \mu'(\alpha_0) = -\frac{\alpha_0 rd(c + d)}{2(c - d)^2} - \frac{rd}{2(c - d)} < 0 \]

Next compute the first Lyapunov coefficient. Fix the parameter \( \alpha \) at its critical value \( \alpha_0 \). At \( \alpha = \alpha_0 \) the nontrivial equilibrium \( E_0 \) has the coordinates
\[ x_1^{(0)} = \frac{d}{c + d}, \quad x_2^{(0)} = \frac{rc}{(c + d)^2}. \]
Translate this to origin by
\[ x_1 = x_1(0) + \xi_1, \quad x_2 = x_2(0) + \xi_2 \]
This transforms system into
\[
\begin{aligned}
\dot{\xi}_1 &= -\frac{cd}{c+d} \xi_1 - \frac{rd}{c+d} \xi_1^2 - c\xi_1 \xi_2 - r\xi_1^3 \equiv F_1(\xi_1, \xi_2) \\
\dot{\xi}_2 &= \frac{rd(c-d)}{(c+d)^2} \xi_1 + (c-d)\xi_1 \xi_2 \equiv F_2(\xi_1, \xi_2)
\end{aligned}
\]
or equivalently
\[
\dot{\xi} = A\xi + \frac{1}{2} B(\xi, \xi) + \frac{1}{6} C(\xi, \xi, \xi)
\]
where \( A = A(\alpha_0) \), and the multilinear functions \( B \) and \( C \) take on the planar vectors \( \xi = (\xi_1, \xi_2)^T \), \( \eta = (\eta_1, \eta_2)^T \) and \( \zeta = (\zeta_1, \zeta_2)^T \):
\[
B(\xi, \eta) = \begin{pmatrix}
-\frac{2rd}{c+d} \xi_1 \eta_1 - c(\xi_1 \eta_2 + \xi_2 \eta_1) \\
(c-d)(\xi_1 \eta_2 + \xi_2 \eta_1)
\end{pmatrix}
\]
\[
C(\xi, \eta, \zeta) = \begin{pmatrix}
-6r\xi_1 \eta_1 \zeta_1 \\
0
\end{pmatrix}
\]
And the matrix
\[
A(\alpha_0) = \begin{pmatrix}
0 & -\frac{cd}{c+d} \\
\omega(c+d) & \omega(c+d)
\end{pmatrix}
\]
where \( \omega^2 = \omega^2(\alpha_0) \). Now we choose one eigenvector as
\[
q = \begin{pmatrix}
\frac{cd}{\omega(c+d)} \\
-\frac{i\omega(c+d)}{cd}
\end{pmatrix}
\]
Then together with \( \langle p, q \rangle = 1 \) we have
\[
p = \frac{1}{2\omega c d (c+d)} \begin{pmatrix}
\omega(c+d) \\
-icd
\end{pmatrix}.
\]
Now we simply calculate
\[
g_{20} = \langle p, B(q, q) \rangle = \frac{cd(c^2 - 2) - rd}{c+d} + i\omega(c+d)^2,
\]
\[
g_{11} = \langle p, B(q, \bar{q}) \rangle = -\frac{rd^2}{c+d}, \quad g_{21} = \langle p, C(q, q, \bar{q}) \rangle = -3r c^2 d^2
\]
Finally we obtain
\[
l_1(\alpha_0) = \frac{1}{2\omega^2} \text{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rd^2 \omega}{\omega^2} < 0
\]
Therefore, the nondegeneracy condition holds. We can now conclude that the system has a unique and stable limit cycle bifurcated from the equilibrium via the Andronov-Hopf bifurcation for \( \alpha < \alpha_0 \).
Example 8 (Andronov-Hopf bifurcation in Brusselator). As an example of a chemical system, we consider the Brusselator. This hypothetical system is composed of substances that react through the following irreversible stages:

\[
\begin{align*}
A & \xrightarrow{k_1} X \\
B + X & \xrightarrow{k_2} Y + D \\
2X + Y & \xrightarrow{k_3} 3X \\
X & \xrightarrow{k_4} E
\end{align*}
\]

Here capital letters denote reagents, while the constants \( k_i \) over the arrows indicate the corresponding reaction rates. The substances \( D \) and \( E \) do not re-enter the reaction, while \( A \) and \( B \) are assumed to remain constant. Thus the law of mass action gives the following system of two nonlinear equations for the concentrations \([X]\), and \([Y]\):

\[
\begin{cases}
\frac{d[X]}{dt} = k_1[A] - k_2[B][X] - k_4[X] + k_3[X]^2[Y] \\
\frac{d[Y]}{dt} = k_2[B][X] - k_3[X]^2[Y]
\end{cases}
\]

Linear scaling of the variables and time

\[
\tau = k_4 t, \quad x_1 = \left(\frac{k_3}{k_4}\right)^{\frac{1}{2}} [X], \quad x_2 = \left(\frac{k_3}{k_4}\right)^{\frac{1}{2}} [Y], \quad a = \frac{k_1}{k_4} \left(\frac{k_3}{k_4}\right)^{\frac{1}{2}} [A], \quad b = \frac{k_2}{k_4} [B]
\]

yields the Brusselator equations,

\[
\begin{align*}
\dot{x}_1 &= a - (b + 1)x_1 + x_1^2 x_2 \equiv F_1(x_1, x_2, a, b) \\
\dot{x}_2 &= bx_1 - x_1^2 x_2 \equiv F_2(x_1, x_2, a, b)
\end{align*}
\]

There is an equilibrium at

\[
E_0 = \left( a, \frac{b}{a} \right)
\]

20
The Jacobian evaluated at this point is

$$A(b) = \begin{pmatrix} b - 1 & a^2 \\ -b & -a^2 \end{pmatrix}$$

whose characteristic polynomial is

$$\lambda^2 - (b - 1 - a^2)\lambda + a^2 = 0.$$  

It is clear that $E_0$ is a stable equilibrium if $b < 1 + a^2$, and unstable if $b > 1 + a^2$. At $b = 1 + a^2$ the system undergoes an Andronov-Hopf bifurcation. Now we are going to investigate if this bifurcation is supercritical or subcritical. We use $b$ as the control parameter. So at $b = b_0 := 1 + a^2$ the equilibrium is $E_0(b_0) = (a, (1 + a^2)/a)$ with purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$, $\omega = a > 0$.

$$\mu(b) = \frac{\sigma(b)}{2} = \frac{b - 1 - a^2}{2}$$

We have $\mu(b_0) = 0$ for $b_0 = 1 + a^2$; and it is easy to check

$$\mu'(b_0) = \frac{1}{2} \neq 0.$$  

Next we compute the first Lyapunov coefficient. To this end we translate the nontrivial equilibrium to the origin by

$$x_1 = a + \xi_1, x_2 = \frac{1 + a^2}{a} + \xi_2$$

This transforms the system to

$$\begin{cases} 
\dot{\xi}_1 = a^2\xi_1 + a^2\xi_2 + (a + 1/a)\xi_1^2 + 2a\xi_1\xi_2 + \xi_2^2 \\
\dot{\xi}_2 = -(1 + a^2)\xi_1 - a^2\xi_2 - (a + 1/a)\xi_1^2 - 2a\xi_1\xi_2 - \xi_2^2 
\end{cases}$$

or equivalently,

$$\dot{\xi} = A\xi + \frac{1}{2}B(\xi, \xi) + \frac{1}{6}C(\xi, \xi, \xi)$$

where

$$A = A(b_0) = A(a + a^2) = \begin{pmatrix} a^2 & a^2 \\ -(1 + a^2) & -a^2 \end{pmatrix}$$

$$B(\xi, \eta) = \begin{pmatrix} 2(a + 1/a)\xi_1\eta_1 + 2a(\xi_1\eta_2 + \xi_2\eta_1) \\ -2(a + 1/a)\xi_1\eta_1 - 2a(\xi_1\eta_2 + \xi_2\eta_1) \end{pmatrix}$$

$$C(\xi, \eta, \zeta) = \begin{pmatrix} 2(\xi_1\eta_1\zeta_2 + \xi_1\zeta_1\eta_2 + \eta_1\zeta_1\xi_2) \\ -2(\xi_1\eta_1\zeta_2 + \xi_1\zeta_1\eta_2 + \eta_1\zeta_1\xi_2) \end{pmatrix}$$

A straightforward computation yields after normalization $\langle p, q \rangle = 1$,

$$q = \begin{pmatrix} -ia + a^2 \\ 1 + a^2 \end{pmatrix}, \quad p = \begin{pmatrix} -\frac{i(1 + a^2)}{2} \\ \frac{1 - 2ia}{1} \end{pmatrix}$$
Now we compute the coefficients till Taylor expansion of $g(z, \bar{z})$. A tedious calculation leads to

$$g_{20} = \langle p, B(q, q) \rangle = a - i, g_{11} = \langle p, B(q, \bar{q}) \rangle = \frac{(a - i)(a^2 - 1)}{1 + a^2}, g_{21} = \langle p, C(q, \bar{q}) \rangle = -\frac{2 + a^2}{2a(1 + a^2)}$$

Finally, the first Lyapunov coefficient evaluated at $b = b_0$ is

$$l_1(b_0) = l_1(1 + a^2) = \frac{1}{2\omega^2} \text{Re}(i g_{20} g_{11} + \omega g_{21}) = -\frac{2 + a^2}{2a(1 + a^2)} < 0.$$ 

Hence, there is a unique stable limit cycle for $b = 1 + a^2$, and the Andronov-Hopf bifurcation is supercritical.

4.3 Andronov-Hopf bifurcation in higher dimensional systems

In the $n$-dimensional case ($n \geq 3$), $\dot{x} = f(x, \alpha)$, $x \in \mathbb{R}^n$ depending on a parameter $\alpha \in \mathbb{R}$, where $f$ is smooth, the Jacobian matrix at $\alpha = 0$, $A(0)$, has

- a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, as well as
- $n_s$ eigenvalues with negative real parts, and
- $n_u$ eigenvalues with positive real parts
with $n = n_s + n_u + 2$. According to the Center Manifold Theorem, there is a family of smooth two-dimensional invariant manifold $W^c_\alpha$ near the origin. The $n$-dimensional system restricted on $W^c_\alpha$ is two-dimensional, hence has the normal form as we have discussed.

Moreover, under the non-degenerary condition (AH.1) and AH.2), the $n$-dimensional system is locally topologically equivalent near the origin to the suspension of the normal form by the standard saddle, i.e.

$$
\begin{align*}
\dot{y}_1 &= \beta y_1 - y_2 + \sigma y_1 (y_1^2 + y_2^2) \\
\dot{y}_2 &= y_1 + \beta y_2 + \sigma y_1 (y_1^2 + y_2^2) \\
\dot{y}^s &= -y^s \\
\dot{y}^u &= y^u
\end{align*}
$$

where $y = (y_1, y_2)^T \in \mathbb{R}^2$, $y^s \in \mathbb{R}^{n_s}$, $y^u \in \mathbb{R}^{n_u}$. The next figure shows a phase portraits of the normal form suspension when $n = 3$, $n_s = 1$, $n_u = 0$, and $\sigma = -1$.

Whether Andronov-Hopf bifurcation is subcritical or supercritical is determined by $\sigma$, which is the sign of the first Lyapunov coefficient $l_1(0)$ of the dynamical system near the equilibrium. This coefficient can be computed at $\alpha = 0$ as follows. Write the Taylor expansion of $f(x, y)$ at $x = 0$

$$
f(x, 0) = A_0 x + \frac{1}{2} B(x, x) + \frac{1}{6} C(x, x, x) + O(||x||^4),$$

where $B(x, y)$ and $C(x, y, z)$ are the multilinear functions with components

$$
B_j(x, y) = \sum_{k, l=1}^{n} \frac{\partial^2 f_j(\xi, 0)}{\partial \xi_k \partial \xi_l} \bigg|_{\xi = 0} x_k y_l
$$

$$
C_j(x, y, z) = \sum_{k, l, m=1}^{n} \frac{\partial^3 f_j(\xi, 0)}{\partial \xi_k \partial \xi_l \partial \xi_m} \bigg|_{\xi = 0} x_k y_l z_m
$$

where $j = 1, 2, ..., n$. Let $q \in \mathbb{C}^n$ be a complex eigenvector of $A_0$ corresponding to the eigenvalues $i\omega_0$:

$$
A_0 q = i \omega q.
$$
Introduce as before an adjoint eigenvector \( p \in \mathbb{C}^n \):

\[
A_0^T p = -i \omega_0 p, \quad \langle p, q \rangle = 1
\]

Then we can derive the first Lyapunov coefficient \( l_1(0) \):

\[
l_1(0) = \frac{1}{2 \omega_0} \text{Re} \left[ \langle p, C(q, q, \overline{q}) \rangle - 2 \langle p, B(A_0^{-1}B(q, q)) \rangle + \langle p, B(2i \omega_0 I_n - A_0)^{-1}B(q, q) \rangle \right]
\]

where \( I_n \) is the \( n \times n \) identity matrix. Note that the value (but not the sign) of \( l_0(0) \) depends on the scaling of the eigenvector \( q \). The normalization \( \langle q, q \rangle = 1 \) is one of the options to remove this ambiguity.

### 4.4 Some important examples

The Lyapunov coefficient can be found easily in some simple but important examples. Here we assume that \( a, b > 0 \) are positive parameters and all derivatives should be evaluated at the critical equilibrium.

<table>
<thead>
<tr>
<th>System</th>
<th>Condition</th>
<th>( \text{sign } l_1(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \dot{x}_1 = F(x_1) - x_2 ) &lt;br&gt; ( \dot{x}_2 = a(x_1 - b) )</td>
<td>( F' = 0 )</td>
<td>( \text{sign } F'' )</td>
</tr>
<tr>
<td>( \dot{x}_1 = F(x_1) - x_2 ) &lt;br&gt; ( \dot{x}_2 = a(bx_1 - x_2) )</td>
<td>( F' = a ) and ( b &gt; a )</td>
<td>( \text{sign}[F'' + (F'')^2/(b - a)] )</td>
</tr>
<tr>
<td>( \dot{x}_1 = F(x_1) - x_2 ) &lt;br&gt; ( \dot{x}_2 = a(G(x_1) - x_2) )</td>
<td>( F' = a ) and ( G' &gt; a )</td>
<td>( \text{sign}[F'' + F''(F'' - G'')/(G' - a)] )</td>
</tr>
</tbody>
</table>

### Exercises

1. Check that each of the following system has an equilibrium that exhibits the Andronov-Hopf bifurcation at some value of \( \alpha \), and compute the first Lyapunov coefficient:

   (a) **Reyleigh’s equation:**
   
   \[
   \ddot{x} + \dot{x}^3 - 2\alpha \dot{x} + x = 0
   \]

   (b) **van der Pol’s oscillator:**
   
   \[
   \dot{y} - (\alpha - y^2)\dot{y} + y = 0
   \]

   (c) **Bautin’s example:**
   
   \[
   \begin{cases}
   \dot{x} = y \\
   \dot{y} = -x + \alpha y + x^2 + xy + y^2
   \end{cases}
   \]

   (d) **Advertising diffusion model**
   
   \[
   \begin{cases}
   \dot{x}_1 = \alpha(1 - x_1x_2^2 + A(x_2 - 1)) \\
   \dot{x}_2 = x_1x_2^2 - x_2
   \end{cases}
   \]
2. Suppose that a system at the critical parameter values corresponding to the Andronov-Hopf bifurcation has the form

\[
\begin{align*}
\dot{x} &= -\omega y + \frac{1}{2} f_{xx} x^2 + f_{xy} x y + \frac{1}{6} f_{xxx} x^3 + \frac{1}{2} f_{xxy} x^2 y + \frac{1}{2} f_{xyy} xy^2 + \frac{1}{6} f_{yyy} y^3 + \cdots \\
\dot{y} &= \omega x + \frac{1}{2} g_{xx} x^2 + g_{xy} x y + \frac{1}{6} g_{xxx} x^3 + \frac{1}{2} g_{xxy} x^2 y + \frac{1}{2} g_{xyy} xy^2 + \frac{1}{6} g_{yyy} y^3 + \cdots
\end{align*}
\]

Compute \( l_1(0) \). (See [1])

3. (Saddle-node bifurcation in ecology) Consider the following differential equation, which models a single population under a constant harvest:

\[
\dot{x} = rx \left( 1 - \frac{x}{K} \right) - \alpha
\]

where \( x \) is the population number; \( r \) and \( K \) are the *intrinsic growth rate* and the *carrying capacity* of the population, respectively, and \( \alpha \) is the *harvest rate*, which is a control parameter. Find a parameter value \( \alpha_0 \) at which the system has a saddle-node bifurcation, and check the genericity conditions. Based on the analysis, explain what might be a result of overharvesting on the ecosystem dynamics. Is the bifurcation catastrophic in this example?

References

