Graph $G = (V, E)$

- $V = \{1, 2, 3, 4, 5, 6, 7\}$

  \[
  \begin{align*}
  &\{1,2\}, \{1,3\}, \{1,6\}, \\
  &\{2,3\}, \{2,5\}, \{2,6\}, \\
  &\{3,4\}, \\
  &\{4,6\}, \\
  &\{5,6\}
  \end{align*}
  \]

- $E = \{3, 4\}$

- Adjacency Matrix
  - An $n \times n$ binary (if the graph is simple) matrix
  - $A(i, j) = 1 \iff \{i, j\} \in E$
Adjacency Matrix

• An $n \times n$ matrix
• Sum of each row (Column) is the degree of the node
Incidence Matrix

• An $n \times e$ matrix,
• Sum of each row indicates the degree of the node
• Sum of each column is 2
Simple Graphs vs Multi-graphs

• A Simple Graph:
  • Between each pair of nodes there is at most one edge
  • There is no loop (edges like \((u, u)\)) in the graph
Weighted Graph

- A Graph $G = (V, E)$,
- Weight function: $w: E \rightarrow \mathbb{R}$
Subgraph (Subset of a Graph)

- A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ s.t.
  - $V' \subseteq V$
  - $E' \subseteq E$
Degree of a Vertex

• **Un-directed** Graph
  The number of edges incident of $v$

• **Directed** Graph
  • Indegree
    • The number of edges that go in to the node
  • Outdegree
    • The number of edges that go out from the node
Directed Graph

• $V = \{1, 2, 3, 4, 5, 6, 7\}$
• $E = \{(1,2), (1,3), (2,4), (3,4), (4,1)\}$

• Source
  • Indegree = 0

• Sink
  • Outdegree = 0
Degree of a Vertex (in an undirected graph)

• Lemma:

\[ \sum_{v \in V} d(v) = 2|E| \]

• Proposition: The number of odd vertices is Even.
A path, tour and walk

• **Walk:** a sequence $v_0, e_1, v_1, e_2, v_2, ..., e_n, v_n$ is a walk (Like random walk)
• **Tour:** a walk such that edges are different
• **Simple Path:** a Tour s.t. nodes are different.
Isomorphic Graphs

• 2 graphs $G = (V, E)$ and $G' = (V', E')$ are Isomorphic ($G \cong G'$) iff there exist

\[
\begin{align*}
\varphi &: V(G) \to V(G') \\
\psi &: E(G) \to E(G')
\end{align*}
\]

\[
(v_1, v_2) \in E \quad \equiv \quad (\varphi(v_1), \varphi(v_2)) \in E'
\]
Example

• Are the following graphs isomorphic?

From the book Graph Theory by J.A. Bondy, U.S.R. Murty
Example

• Prove that the following graphs are isomorphic

From the book Graph Theory by J.A. Bondy, U.S.R. Murty
Example (Answer)

• Prove that the following graphs are isomorphic

Thanks to Anna for her effort
Complete Graphs

• A simple undirected graph that includes all the possible edges.

• The complete graph on $n$ nodes: $K_n$

• The number of edges in the graph: $\binom{n}{2}$
Complement of a Graph

• For any given graph $G = (V, E)$ the complement of the G is
  o A graph $G^C = (V, E^C)$
  o $(V, E \cup E^C) = K_{|V|}$
Clique

• A subset of a graph (subgraph) that is complete.

• Give an algorithm that for a given constant $k$ and a given graph, finds whether the graph has a $k$-clique or not.
Trees and Forests

• Tree: A connected graph without any cycles
  • $|E| = |V| - 1$

• Forest: A graph without any cycle
Application and Famous Examples
Childhood Challenges

- Whether we could draw this shapes without taking our hands out from the paper, and go through each edges once.
Childhood Challenges

• Which of the following shapes you can draw without taking your hand from the paper, also go through each edge exactly once?

From the book Graph Theory by J.A. Bondy, U.S.R. Murty
Dear ..... (I do not remember you name),
Thanks a lot for your effort. Now you know that although graph 3 looks better, it is not the answer, cause you have tried that.. Hope you are not mad at me.. The answer is the tricky one... Graph 1 and the answer is shown.

From the book Graph Theory by J.A. Bondy, U.S.R. Murty
Water pipeline (Flow Network)

- Each pipeline has a capacity, indicated by the diameter.
- There is a source.
- There is a sink.
- The max amount of water that goes out from the sink in each time (like min).

M. Fragiadakis et al. 2013
Seven Bridges of Königsberg

- The river divides the city into 4 parts
- There are 7 bridges
- People tried to go from each bridge only one time.
- Come back to the starting point

Picture from wikipedia
Eulerian cycle/tour

• EC:
  • A cycle
  • Starts from a node
  • Visits all the edges just one time
  • Come back to the first node
Eulerian cycle/tour

• ET:
  • A Path
  • Starts from a node
  • Visits all the edges just one time
  • Come back to the first node
Eulerian cycle/tour

• Theorem:

A connected graph is Eulerian (contains an Eulerian cycle) iff there is no odd nodes in the graph.

Proof:
1) If it has an Eulerian cycle:
2) If all the nodes have even degree:
Eulerian cycle/tour

- Now come back to the previous problems:
Travelling Salesman Problem (TSP)

• A salesman starts from a city
• Wants to visit all the cities
• Comes back to the main city
Hamiltonian Path/ Cycle

• Hamiltonian path for a given pair of nodes $u$ and $v$:
  A path that visits all the nodes of the graph only one time.

• Hamiltonian cycle:
  A cycle that visits all the nodes of the graph only one time.

Picture from Wikipedia
Multi-partite Graphs
Bipartite Graphs

• A graph $G = (V, E)$, such that we can divide nodes into two sets $V', V''$ s.t.
  • There is no edge between nodes in $V'$ nor between nodes in $V''$
Bipartite Graphs

• Partition:

• Dividing nodes into two (or some) parts s.t.

1. The intersection of each two parts is 0
   • $P_i \cap P_j = 0, \ \forall \ i, j$

2. The union of them is all the nodes
   • $\cup P_i = V$
Bipartite Graphs

- A graph $G = (V, E)$, such that we can divide nodes into two sets $V', V''$ s.t.
  - There is no edge between nodes in $V'$ nor between nodes in $V''$
r-Partite Graph

• For $r \geq 2$ a graph $G = (V, E)$ is called $r$–partite if we can partition nodes $V$ into $P_1, P_2, ..., P_r$ s.t.
  • There is no edges between nodes in $P_i$ for $i \in \{1, ..., r\}$
  • (Vertices in one class should not be adjacent.)
Complete r-partite graphs

• An $r$ – partite graph that includes all the edges is called complete.
• $K_{3,3}$

• $K_{2,2,2}$
• **Theorem:** A graph is bipartite if and only if it contains no odd cycle.

1. If a graph is bipartite: (contradiction if it has an odd cycle)

2. If a graph does not have any odd cycles: (you can write an algorithm to partition the nodes into two groups)
Matching
Matching

- A set of independent edges in a graph (without same vertices).
Maximum Matching Problem

• What is the size of the maximum matching in a given graph?

• Algorithm (out of this course)

  **Input:** An unweighted and underacted graph $G = (V, E)$.
  **Output:** A matching $M$ in the graph $G$ with maximum value $|M|$. 

Reference

• http://www.zib.de/groetschel/teaching/WS1314/BondyMurtyGTWA.pdf


• https://sidiropo.people.uic.edu/courses/2017_spring_math8500_graphalg/scribe/lecture7_maximum_matching.pdf