1.1 Excess of Loss Reinsurance

A simple Excess of loss reinsurance contract was introduced in Example 1.3 in Johansson. In this section we will expand this example to more complicated contracts and discuss their pricing.

Reinsurance is introduced in order to reduce the risk for the primary insurance company, called the cedant. Basically, (per claim) excess of loss reinsurance is defined for individual claims — as opposed to Stop loss contracts which target the aggregate cost. Typically, the reinsurance cover is split into several layers. For example, the notation $10 \times 30$ is used to denote the fact that the reinsurer pays that part of the claim cost that exceeds 30, but with a total liability of at most 10 (in some monetary unit). The part of the cost that exceeds 40 may be covered by other reinsurance contracts, possibly split into several layers; if not, it has to be payed by the cedant and is called spill-over. Each claim cost can then be decomposed as indicated in Table 1.1. The cedant decides the amount of risk that is to be kept by the company – the underlying retention – by choosing the excess point $r_1$. A reasonable choice for the upper limit $r_4$ is the (by judgement) estimated maximum possible/probable loss. In the example of Table 1.1, three reinsurers share the cover $(r_1, r_4)$ with one layer each.

<table>
<thead>
<tr>
<th>Layer no.</th>
<th>Interval</th>
<th>Cover</th>
<th>Payed by</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(0, r_1)$</td>
<td>Retention</td>
<td>Cedant</td>
</tr>
<tr>
<td>1</td>
<td>$(r_1, r_2)$</td>
<td></td>
<td>Reinsurer no. 1</td>
</tr>
<tr>
<td>2</td>
<td>$(r_2, r_3)$</td>
<td></td>
<td>Reinsurer no. 2</td>
</tr>
<tr>
<td>3</td>
<td>$(r_3, r_4)$</td>
<td></td>
<td>Reinsurer no. 3</td>
</tr>
<tr>
<td>4</td>
<td>$(r_4, \infty)$</td>
<td>Spill-over</td>
<td>Cedant</td>
</tr>
</tbody>
</table>

Table 1.1: Schematic overview of excess of loss cover of large claims.

Consider a single, arbitrary contract $\ell \times r$ reinsurance. A benchmark for the pricing of such a contract is the pure premium (or risk premium), equal to the expectation of the reinsurer’s claim costs. In the derivation of the pure premium we will use the collective model, where $N$ is the number of non-zero claims and $X_1, X_2, \ldots, X_N$ are iid individual claim costs, independent of $N$. The layer’s part of the cost of a single claim is

$$
\min(X_i, r + \ell) - \min(X_i, r) = \begin{cases} 
0 & \text{if } X_i \leq r, \\
X_i - r & \text{if } r < X_i < r + \ell, \\
\ell & \text{if } X_i \geq r + \ell.
\end{cases}
$$

Thereby the total cost for the layer is

$$
S = \sum_{i=1}^{N} (\min(X_i, r + \ell) - \min(X_i, r))
$$

We let $L_X(a)$ denote the expectation function for $X_i$, $L_X(a) = E[\min(X_1, a)]$. By taking expectations in (1.1) we find that

$$
E[S] = E[N] (L_X(r + \ell) - L_X(r))
$$

For the estimation of this pure premium, suppose that we have fitted a POT model to the data above a capture limit $c$, see Section 5.3 in Johansson. The capture limit $c < r$ should as usual be chosen low
enough to enable sufficient precision in the parameter estimates, but at the same time large enough to make the Pareto distribution fit the data well. After estimating the parameters, we can recalculate the Poisson parameter to the variable giving the number of claims with cost above any particular excess level \( r \). From now on \( N \) denotes this variable and \( N \sim Po(\lambda) \). We can also recalculate the Pareto parameters to \( X_i \), now defined as the excesses over the level \( r \), i.e. if \( \tilde{X}_i \) is the claim cost considered above, then \( X_i = \tilde{X}_i | \tilde{X}_i > 0 \). Thus we have \( X_i \sim Pa(\alpha, \gamma) \), where the parameters are calculated from the values estimated in the POT model.

Then the cost for the layer simplifies to

\[
S = \sum_{i=1}^{N} \min(X_i, \ell)
\]

and the pure premium is

\[
E[S] = E[N] L_X(\ell) = \lambda \frac{\alpha}{\gamma - 1} \left( 1 - \left( \frac{\alpha}{\alpha + \ell} \right)^{\gamma - 1} \right)
\]

where \( L_X \) was derived in Johansson, Section 3.1.

**1.1.1 Aggregate Deductible**

With an *aggregate deductible* \( A \), the reinsurer pays only the part of the cost \( S \) that exceeds \( A \). The rationale behind this is that the cedant may have capacity to cover a few large claims, but not too many. The cost for \( \ell \) xs \( r \) with aggregate deductible \( A \) is

\[
S_A = S - \min(S, A) = \begin{cases} 
0 & \text{if } S \leq A, \\
S - A & \text{if } S > A,
\end{cases}
\]

where \( S \) is defined in (1.3). The pure premium is thus

\[
E[S_A] = E[S] - L_S(A)
\]

Under the Pareto assumption of the POT model, the excesses over \( r \) have the distribution function \( F_{X_i}(x) = 1 - \left( \frac{\alpha}{\alpha + x} \right)^{\gamma} \). Hence, the distribution function for \( Z_i = \min(X_i, \ell) \) is

\[
G_{Z_i}(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 - \left( \frac{\alpha}{\alpha + x} \right)^{\gamma} & \text{if } 0 < x < \ell; \\
1 & \text{if } x \geq \ell.
\end{cases}
\]

Note the point mass \( P(\min(X_i, \ell) = \ell) = \left( \frac{\alpha}{\alpha + \ell} \right)^{\gamma} \). In order to compute the pure premium of this contract, we need \( L_S(A) \). Hence we need the full distribution of \( S \), even if we are only interested in the expectation of the cost. To this end we use Panjer’s recursion formula. This requires discretising the continuous part of \( G_{Z_i} \). A simple way to do this is the following.
First we choose some step length $h$, and for simplicity we take an $h$ such that $m = \ell/h$ is an integer.

The shorter the step length, the better the approximation, but the longer the running time for Panjer’s recursion. For simplicity in the formulas, we use the rounding method: all claim amounts are rounded to the closest multiple of $h$. This yields the discrete probability distribution

\[
\begin{align*}
f_0 &= G(h/2), \\
f_k &= G(kh + h/2) - G(kh - h/2), \quad k = 1, 2, \ldots, m - 1, \\
f_m &= 1 - G(\ell - h/2).
\end{align*}
\]

Now Panjer’s recursion algorithm yields the probability distribution $g_k = P(S = k)$ for $k = 0, 1, 2, \ldots$. For simplicity in notation, we assume that $A/h$ is also an integer, but this is not really necessary for the computations.

\[
L_S(A) = E[\min(S, A)] = \sum_{k=0}^{A/h-1} kh g_k + A \sum_{k=A/h}^{\infty} g_k = A - \sum_{k=0}^{A/h-1} (A - kh) g_k.
\]

### 1.1.2 Free Reinstatements

Low layers, which have a substantial claims frequency each year, are called *working layers*. For such layers, the reinsurer gets a good estimate of the pure premium as outlined above. High layers, on the other hand, may only occasionally be hit by a claim; such levels are sometimes called *catastrophe layers*. Here, the situation for the reinsurer is more insecure, and she therefore imposes an overall limit to the cost, by only allowing a fixed number of reinstatements of the cover.

Reinstatement, after a claim, means that the reinsurance is reset to give complete cover of the layer $(r, r + \ell)$, for the next claim hitting the layer during the same year. Often this requires extra premium; we start by looking at the situation with *free reinstatements*, though. With $K$ reinstatements, the payments are limited to $(K + 1)\ell$, for some integer $K \geq 0$. The cost for the reinsurer is now

\[
S^K_A = \min(S; A + (K + 1)\ell) - \min(S; A) = \begin{cases} 
0 & \text{if } S \leq A, \\
S - A & \text{if } A < S < A + (K + 1)\ell, \\
(K + 1)\ell & \text{if } S \geq A + (K + 1)\ell.
\end{cases}
\]

Note that we allow the special cases $K = 0$ (no reinstatements) and $K = \infty$ (no limit to the cover). Upon taking expectations, we get the pure premium for $\ell$ $xs$ $r$ with aggregate deductible $A$ and $K$ reinstatements:

\[
E[S^K_A] = L_S(A + (K + 1)\ell) - L_S(A).
\] (1.4)

Note the similarities with Equation (1.2). Imposing the aggregate deductible $A$ and upper limit $A + (K + 1)\ell$ on $S$ is just like having a $(K + 1)\ell$ $xs$ $A$ restriction at the aggregate level.

**Exercise 1.1** A Stop loss (S/L) reinsurance targets the total cost for the aggregate claims, without reference to the individual claims. The reinsurer pays a share $c$, e.g. 95 %, of the amount of the aggregate cost that exceeds some threshold $A$. 

3
If $c = 1$, then S/L is a particular case of $\ell x s r$ with aggregate deductible $A$ and $K$ reinstatements. What choice of the parameters $\ell$, $r$, $A$ and $K$ results in an S/L reinsurance? What is the pure premium in this case?

Note: The S/L contract can also have a limited cover, with an upper limit $B$ to the amount paid by the reinsurer.

### 1.1.3 Paid Reinstatements

 Often, the reinsurer charges an extra premium for reinstating the full cover $\ell$. The reinstatement premium is expressed as a percentage of the initial premium $P$, so that reinstatement $k$ costs $c_k P$, for some $c_k \geq 0$. (Typically, $c_k$ is given as a percentage between 0 % and 100 %.) If a claim does not use the entire cover $\ell$, i.e. $X_i < r + \ell$, then the reinstatement premium is paid pro rata of the claims, i.e. in proportion to the cost of the reinsurer.

The cost of the $k$:th reinstated cover is

$$Y_k = \min(S, A + k\ell + \ell) - \min(S, A + k\ell) = \begin{cases} 0 & \text{if } S \leq A + k\ell, \\ S - A - k\ell & \text{if } A + k\ell < S < A + k\ell + \ell, \\ \ell & \text{if } S \geq A + k\ell + \ell. \end{cases}$$

Here, we let $k = 0$ denote the initial cover of the aggregated interval $(A, A + \ell)$. The random premium for the $k$:th reinstatement, $k = 1, 2, \ldots, K$, paid pro rata of the claims can now be written

$$\frac{Y_{k-1}}{\ell} c_k P$$

with expectation

$$\frac{1}{\ell} c_k P \cdot \left\{L_S(A + k\ell + \ell) - L_S(A + k\ell)\right\}$$

so that the expected total premium is

$$P_{\text{tot}} = P \left(1 + \frac{1}{\ell} \sum_{k=1}^{K} c_k \left\{L_S(A + k\ell + \ell) - L_S(A + k\ell)\right\}\right)$$

By equating this to the total cost in (1.4),

$$L_S(A + (K + 1)\ell) - L_S(A)$$

we finally get the pure premium for $\ell x s r$ with aggregate deductible $A$ and $K$ paid reinstatements.

Esbjörn Ohlsson, 2005-09-30

Note: This is a preliminary version not for use outside the courses at Stockholm University.
### 1.2 Liten ordlista

Tabell 1.2: Engelska och svenska termer i återförsäkring

<table>
<thead>
<tr>
<th>Engelska Term</th>
<th>Svenska Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate deductible</td>
<td>Aggregerat självbehåll</td>
</tr>
<tr>
<td>Cedant</td>
<td>Cedent</td>
</tr>
<tr>
<td>Claim</td>
<td>Skada</td>
</tr>
<tr>
<td>Deductible</td>
<td>Självrisk (direktförsäkring)</td>
</tr>
<tr>
<td>Excess of loss</td>
<td>Excess of loss</td>
</tr>
<tr>
<td>Layer</td>
<td>Layer</td>
</tr>
<tr>
<td>Pure premium</td>
<td>Riskpremie</td>
</tr>
<tr>
<td>Reinstatement</td>
<td>Reinstatement</td>
</tr>
<tr>
<td>Reinsurance</td>
<td>Återförsäkring</td>
</tr>
<tr>
<td>Retention</td>
<td>Självbehåll (återförsäkring)</td>
</tr>
<tr>
<td>Stop Loss</td>
<td>Stop Loss</td>
</tr>
</tbody>
</table>