MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II (Online course, omtentamen)
August 20, 2014
9:00 - 14:00

The exams can be picked up from Katarina Ringels (Hus 6, Rum 204).

Complete and clear solutions must be given
except where explicitly stated otherwise.

No calculators allowed.
If you cannot simplify an expression any further, just leave it.

1. (3 points) Give the rook polynomial of a rectangular 6×13 'chessboard' where one of the corner tiles is removed (so the board has $6 \cdot 13 - 1$ many tiles). You must show your reasoning.

2. (4 points) Find a closed formula for the sequence a_n that satisfies $a_0 = 0$, $a_1 = 67/5$, and for $n \geq 2$ the recursion relation

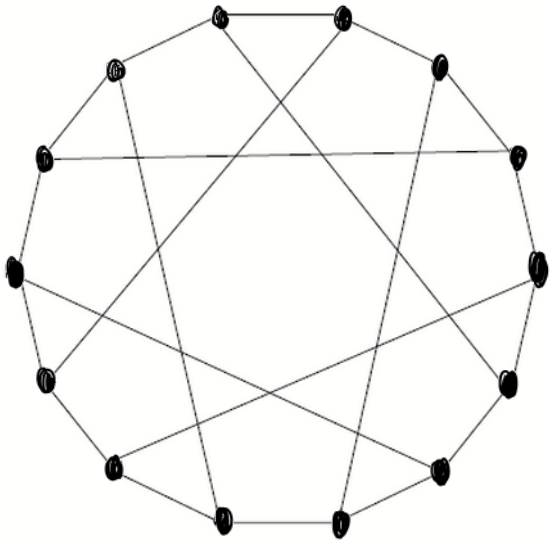
$$a_n - a_{n-1} - 6a_{n-2} = 30 \cdot n \cdot 3^{n-2}.$$

Clearly present every step of your computation.

3. (3 points) Let G be a complete 6-ary rooted tree with 131 leafs. How many vertices does G have?

You have to give a complete argument, don't just use some formula.

4. (3 points)
Prove that the following graph is not planar:



Hint: Kuratowski's theorem. Don't give up too early, this is something to play around with.

5. (3 points) You probably heard about the following fact: every natural number n has a unique representation as a sum of nonnegative integers in base 3. This means that

$$n = c_0 + c_1 \cdot 3 + c_2 \cdot 3^2 + \cdots ,$$

where c_i is either 0, 1 or 2; and these numbers are unique. In this way, $100 = 1+0\cdot 3+2\cdot 9+1\cdot 81$. We say that the decimal number 100 equals **1201** as a *ternary number*.

Here's your problem. How many ternary numbers of length 15 are there consisting of five 0's, five 1's, and five 2's as digits, if it is not allowed that the same digits all are next to each other? (So, not all five 0's are allowed to appear consecutively as digits, and the same holds for the five 1's, and for the five 2's). For instance, **222200010211110** is one of these numbers.

To make your life easier, the first digit is allowed to be 0, so **022220010211110** can be regarded as a ternary number of length 15.

6. (1,5 points) Let a_r be the number of nonnegative integer solutions of the equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = r.$$

Write down *two* different interpretations of a_8 in terms of (unordered) partitions. Use precise language. No justifications necessary.

7. (3 points) 64 employees should be assigned to work on 18 projects. Each employee should work on exactly one project, and for every project at least 3 and at most 4 employees should be assigned. In how many ways can this be done?

As usual it is enough to write down an explicit formula, you don't have to evaluate it.

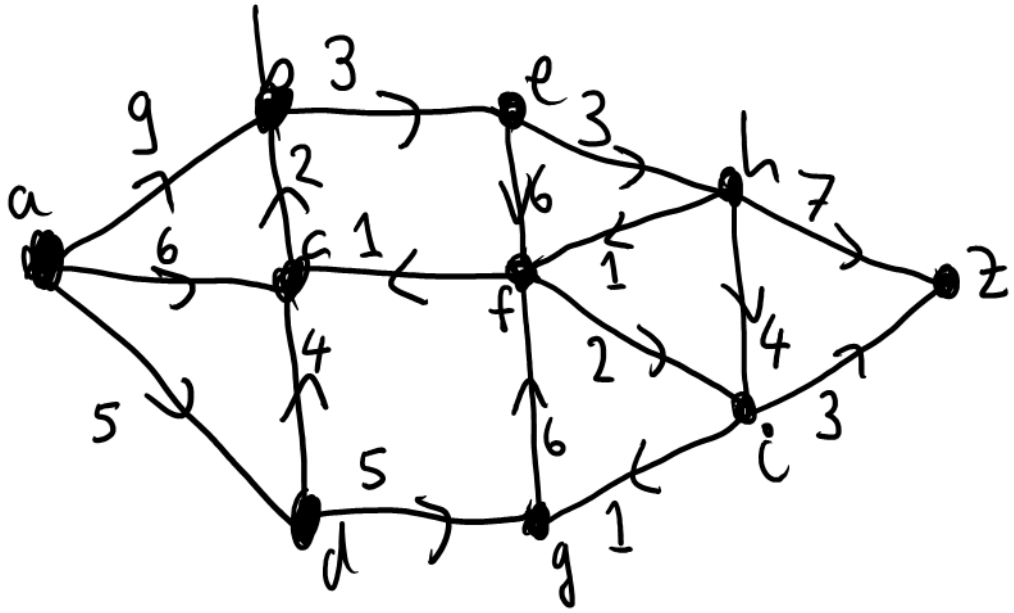
8. (3 points) In a candy factory, surprise boxes containing five types of godis are manufactured. Here the following rules must be followed:

- the number of cola bottles in a box must be divisible by 4,
- the number of slickepinnar in a box must be even,
- the square root of the number of vingummi must be an even integer,
- there cannot be more than three praliner in a box,
- there cannot be more than one chocoladkaka in a box,
- the total number of godis is between 17 and 25.

As you see it is possible that a box might have only one type of godis. Now, during one production cycle every possible kind of box gets produced one after another without ever repeating an assortment. If each assorting of a box takes 5 seconds and you would like to have a box containing a vingummi, what's the longest time you would possibly have to wait?

LAST PROBLEM ON NEXT PAGE.

9. (6,5 points) Consider the following network:



(a) (2,5 points) Use Dijkstra's algorithm to find for any vertex $v = a, b, c, d, e, f, g, h, i, z$ the distance $d(a, v)$ from a to v.

You MUST USE Dijkstra's algorithm in order to get any credits. Clearly show your table and in which order you choose the vertices (use an extra page, since you will need some space).

(b) (1 points) Give a shortest directed path from a to z.

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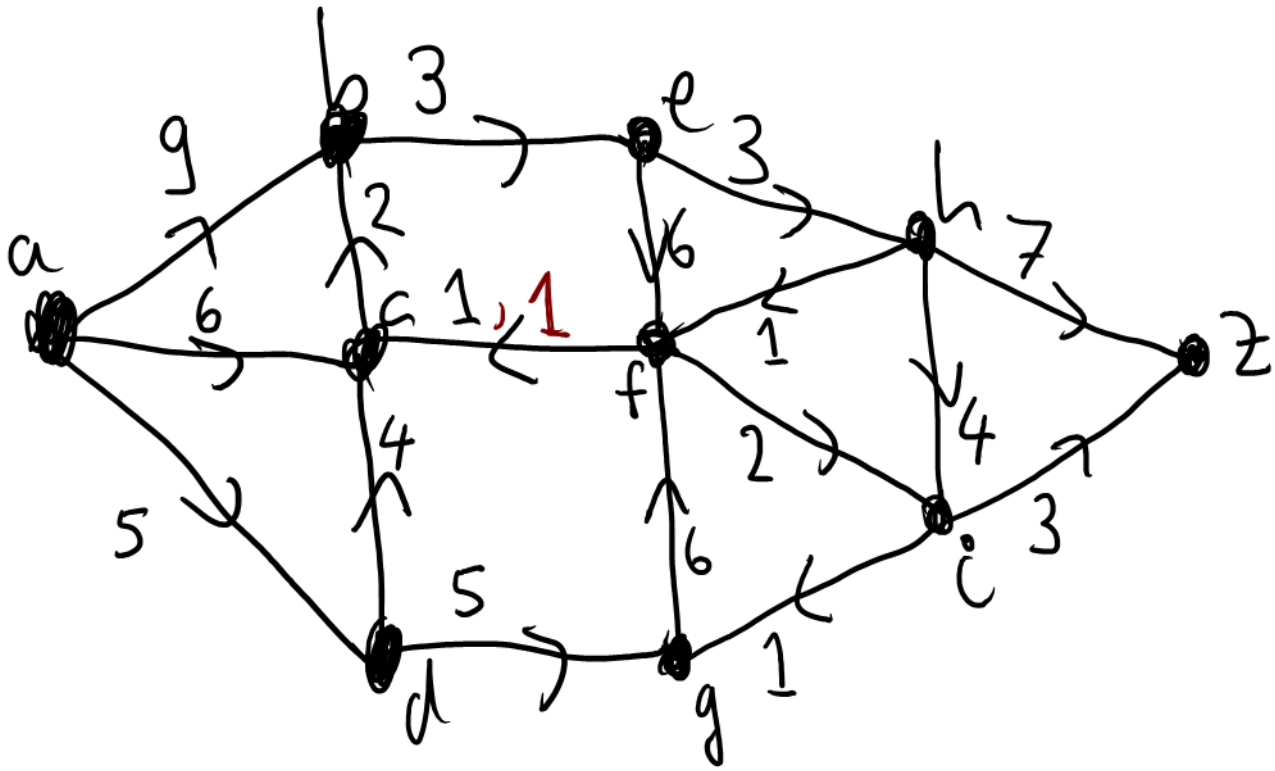
HAND IN THIS PAGE

(c) (2 points) Give a flow with maximal flow value where (to make things more interesting) *the arrow from f to c has flow value* 1.

Enter the flow you found RIGHT HERE into the network (next to the capacities):



Write here the flow value of your flow:

(d) (1 points) Give a cut with the minimal cut capacity RIGHT HERE:

Write here the cut capacity of your cut: