MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II (Online course)
May 26, 2015
9:00 - 14:00

SOLUTIONS

Complete and clear solutions must be given
except where explicitly stated otherwise.

No calculators allowed.
If you cannot simplify an expression any further, just leave it.

1. (3 points) During a dinner at a conference in Combinatorics, $2n$ mathematicians are seated around a circular table. They notice that the table has n numbered black seats and n numbered white seats, where black and white seats alternate (so, a black seat has two white seats as neighbors). Of course, being mathematicians they wonder how many possibilities there are to come tomorrow again together at this table so that everyone's chair has the same color as today but now everone sitting on a black chair has a different left neighbor than today. Can you find a precise answer to their question? Can you even give a nice approximation to your answer when n is really large?

Solution:

Let c_i be the condition that the left neighbor of the ith black chair is seated by the same person as yesterday. Then inclusion-exclusion yields the answer

$$\bar{N} = S_0 - S_1 + S_2 - S_3 \pm \cdots \pm S_n,$$

where $N = S_0 = n!n!$ (product rule, and note that the chairs are numbered), and $S_i = \binom{n}{i} n!(n-i)!$. Simplifying yields

$$(n!)^2 \sum_{i=0}^{n} (-1)^i \frac{1}{i!}.$$

Note that this is just $n!$ times the number of derangements of $\{1, \ldots, n\}$ (as one could have seen directly). Hence, this approximates

$$\frac{(n!)^2}{e}.$$

2. (4 points)

(a) (1 point) Give a recursive formula for the number of words of length n over the alphabet $\{0, 1, 2, 3\}$ that have an even number of 0's. (You don't have to solve the recurrence.)

Solution:

Let a_n (resp. b_n) be the number of words of length n over the alphabet $\{0, 1, 2, 3\}$ that have an even (resp. odd) number of 0's. Then if the last letter is in 0, we get b_{n-1} many such words, and if the last letter is in $\{1, 2, 3\}$, then we get $3a_{n-1}$ many words. Since $a_{n-1} + b_{n-1} = 4^{n-1}$, this yields

$$b_{n-1} + 3a_{n-1} = 4^{n-1} - a_{n-1} + 3a_{n-1} = 4^{n-1} + 2a_{n-1}.$$

(b) (3 points) Find a closed formula for the sequence a_n that satisfies $a_0 = 1$, $a_1 = 3$, and for $n \geq 2$ the recursion relation

$$a_n - 4a_{n-2} = -n \cdot 2^{n+1}.$$

Clearly present every step of your computation.

Solution:

We get $a_n^{(h)} = c_1 2^n + c_2(-2)^n$. We guess $a_n^{(p)} = n(An + B)2^n$. Plugging it into the recursion relation yields $A = -1/2, B = -1$. Hence, $a_n = c_1 2^n + c_2(-2)^n - n(n/2+1)2^n$. Plugging in the initial conditions yields $c_1 = 2, c_2 = -1$. This gives the answer

$$2^{n+1} - (-2)^n - n(n/2 + 1)2^n.$$

3. (2 points) In a previous exam we already used the following important mathematical fact: *every natural number n has a unique representation as a sum of nonnegative integers in base 3.* This means that

$$n = c_0 + c_1 \cdot 3 + c_2 \cdot 3^2 + \cdots ,$$

where c_i is either 0, 1 or 2; and these numbers are unique. For instance, $100 = 1+0{\cdot}3+2{\cdot}9+1{\cdot}81$.

Prove above fact by using generating functions. You must use this technique for getting credits. (Hint: Let a_n be the the number of representations of n as a sum of nonnegative integers in base 3. It may help if you think about what the answer is you would like to have, namely, $a_n = ?$)

Solution:

The generating function $\sum_{i=0}^{\infty} a_n x^n$ is the infinite formal product

$$(1 + x + x^2)(1 + x^3 + (x^3)^2)(1 + (x^{3^2}) + (x^{3^2})^2) \cdots = \frac{1 - x^3}{1 - x} \frac{1 - (x^3)^3}{1 - (x^3)} \frac{1 - (x^{3^2})^3}{1 - (x^{3^2})} \cdots$$

which equals after canceling out

$$\frac{1}{1 - x} = \sum_{i=0}^{\infty} x^i.$$

Hence, $a_n = 1$ for all n. (If you don't like infinite products, you can also do it with finitely many products, e.g., n many; then the final sum will have an upper bound.)

4. (2 points) Prove the formula for the number of surjective functions from a set with m elements onto a set with n elements by using the technique of exponential generating functions. You must use this method otherwise you won't get any credits.

Solution:

We are looking for the coefficient of $x^m/m!$ in

$$(x + x^2/2! + \cdots)^n = (e^x - 1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} e^{ix},$$

which equals

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} i^m = n! S(m, n).$$

5. (3,5 points)

- (1 point) You put b numbered balls in u unnumbered urns (some may be empty). How many outcomes are possible?

  Solution:

  $S(b,1) + \cdots + S(b,u)$.

- (2,5 points) Benjamin buys ice cream for Juliane, little Gustav, and himself at the ice cream shop that has 20 different sorts to choose from. Benjamin always likes to have an even number of scoops for each sort of ice cream he is going to choose. Juliane doesn't like mango ice cream and wants at least two scoops of stracciatella. Little Gustav is only allowed to have at most two scoops of ice cream, and no pistachio. If Benjamin buys 10 scoops of ice cream in total for them all, how many possibilities are there?

  You don't have to (and shouldn't) compute the exact number. Instead, I would like to see a precise mathematical expression.

  Solution:

  Wow, I was surprised how many possible misunderstandings and interpretations one can see. Here is how I intended it to be: "at most two scoops of ice cream" meant that little Gustav shouldn't eat more than two scoops overall. "buys 10 scoops of ice cream in total for them all" meant not necessarily that everone of the three had to get some ice cream, it could also happen that someone didn't want any ice cream. "how many possibilities are there" meant for instance that one possibility would be that Benjamin gets two scoops of stracciatella and two scoops of pistaccio, Juliane two scoops of stracciatella and one scoop of lemon, and Gustav just one scoop of chocolate. In other words, I didn't assume that any order matters, neither when ordering the ice cream at the shop nor when stacking the scoops (I should have mentioned that they order the scoops in cups). Okay, now assuming all these solutions, the easiest way is indeed the direct one. Benjamin has $\binom{20-1+r}{r}$ many choices to choose $2r$ many scoops, Juliane has $\binom{19-1+r-2}{r-2}$ many choices to choose r scoops, and Gustav has 1, 19, or $\binom{19-1+2}{2}$ many choices to choose 0, 1, or 2 many scoops of ice cream. Using these hints, I leave it to the reader to figure out the precise answer. (Here, it actually doesn't get easier with generating functions).

6. (6 points) Let $n \geq 3$. Take the complete graph K_n on n vertices and remove an edge E to get the graph G_n.

(a) (1 point) For which n does G_n have an Eulerian circuit?

Solution:

G_n has an Eulerian circuit if and only if every vertex degree is even. However, a vertex of E has degree $n-2$, while a vertex not in E has degree $n-1$. So, G_n has never an Eulerian circuit.

(b) (1,5 points) How many Hamiltonian cycles does G_n have?

Solution:

There are $(n-1)!$ many directed Hamiltonian cycles of K_n, so $(n-1)!/2$ many (undirected) Hamiltonian cycles of K_n. There are $(n-2)!$ many directed Hamiltonian cycles of K_n using the arrow $A \to B$, so also $(n-2)!$ many (undirected) Hamiltonian cycles of K_n that use the edge E. Hence, the answer is $(n-1)!/2-(n-2)!$.

(c) (2 points) Let a_n be the number of edges of G_n. Write the generating function of the sequence a_n as a quotient of two polynomials.

Solution:

$$\sum_{n=3}^{\infty}(\binom{n}{2}-1)x^n = \sum_{n=2}^{\infty}(\binom{n}{2}-1)x^n = x^2\left(\sum_{n=0}^{\infty}(\binom{n+2}{2}-1)x^n\right)$$

$$x^2\left((\sum_{n=0}^{\infty}(\binom{n+2}{2})x^n) - (\sum_{n=0}^{\infty} x^n)\right) = x^2(\frac{1}{(1-x)^3} - \frac{1}{1-x}) = \frac{x^2(1-(1-x)^2)}{(1-x)^3}.$$

(d) (1,5 points) Compute the chromatic polynomial of G_n.

Solution:

We use Theorem 11.10 of the textbook:

$$P(G_n,\lambda) = P(K_n,\lambda) + P(K_{n-1},\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1) + \lambda(\lambda-1)\cdots(\lambda-n+2)$$
$$= \lambda(\lambda-1)\cdots(\lambda-n+3)(\lambda-n+2)^2.$$

7. (2,5 points) Let G be a planar, connected, loop-free graph. Prove that if every vertex of G has at least degree five, then G has at least 12 vertices.

Solution:

We know

$$e \le 3v - 6$$

and

$$5v \le \sum_{v\in V} \deg(v) = 2e.$$

This implies $v \ge 12$.

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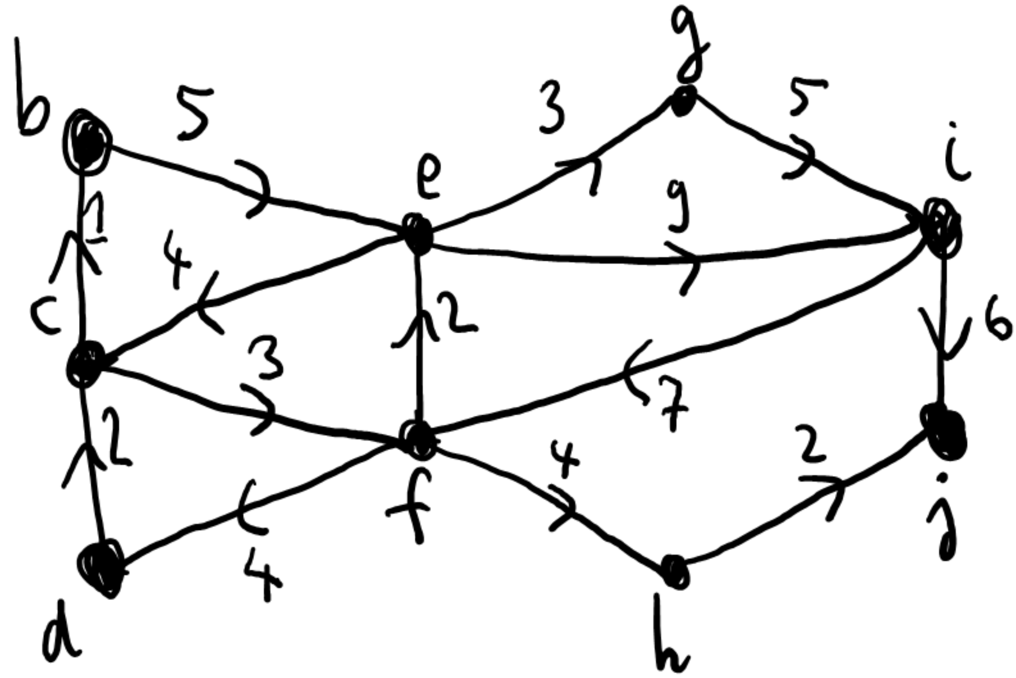
8. (3 points)
Consider the following network:



(a) (2 points) Use Dijkstra's algorithm to find for any vertex $v = b, c, d, e, f, g, h, i, j$ the distance $d(b, v)$ from b to v.

You MUST USE Dijkstra's algorithm in order to get any credits. Clearly show your table and in which order you choose the vertices (use an extra page, since you will need some space).

Here is the table:

c	∞	9	$\underline{9}$					
d	∞	∞	∞	∞	16	$\underline{16}$		
e	$\underline{5}$							
f	∞	∞	∞	$\underline{12}$				
g	∞	$\underline{8}$						
h	∞	∞	∞	∞	16	16	$\underline{16}$	
i	∞	14	13	13	$\underline{13}$			
j	∞	∞	∞	∞	∞	19	19	$\underline{18}$

At one point I could have chosen h instead of d.

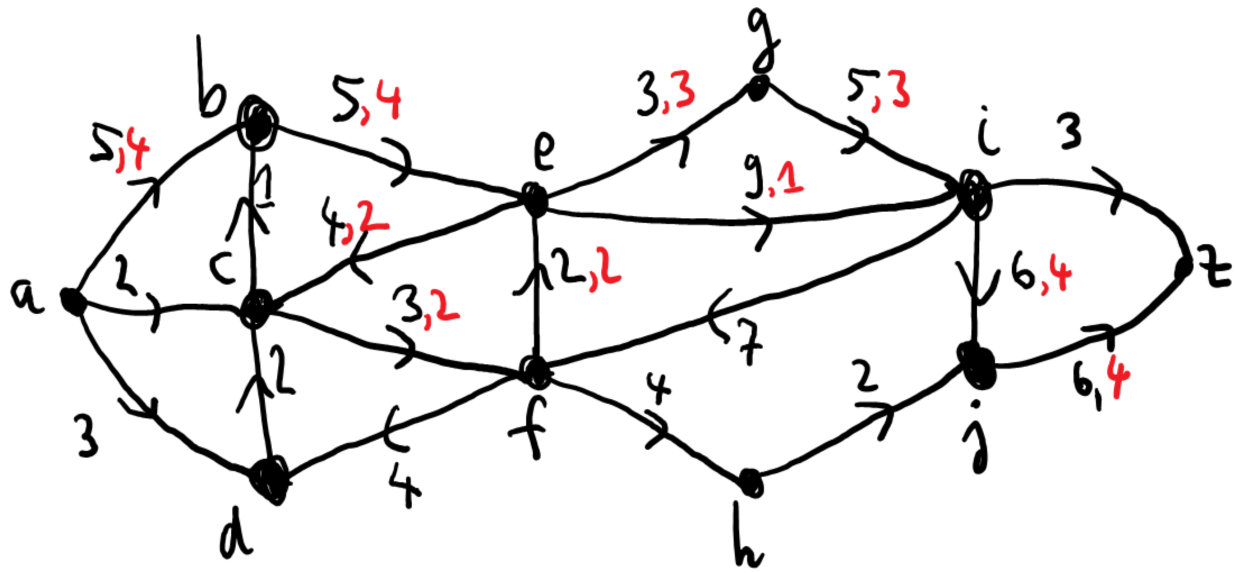
(b) (1 points) Give a shortest directed path from b to j.

Solution:

b, e, c, f, h, j.

9. (4 points)

(a) (3 points) Use the Ford-Fulkerson and Edmonds-Karp algorithms to find a maximal flow for the following network. You MUST start with the given flow (here, the first number next to an edge is its capacity, the second is its flow value - if there is no second number it is 0).



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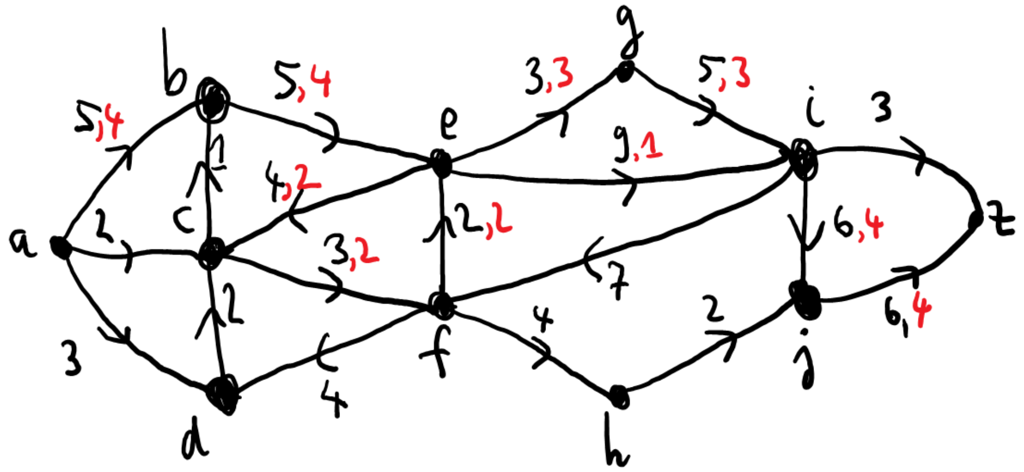
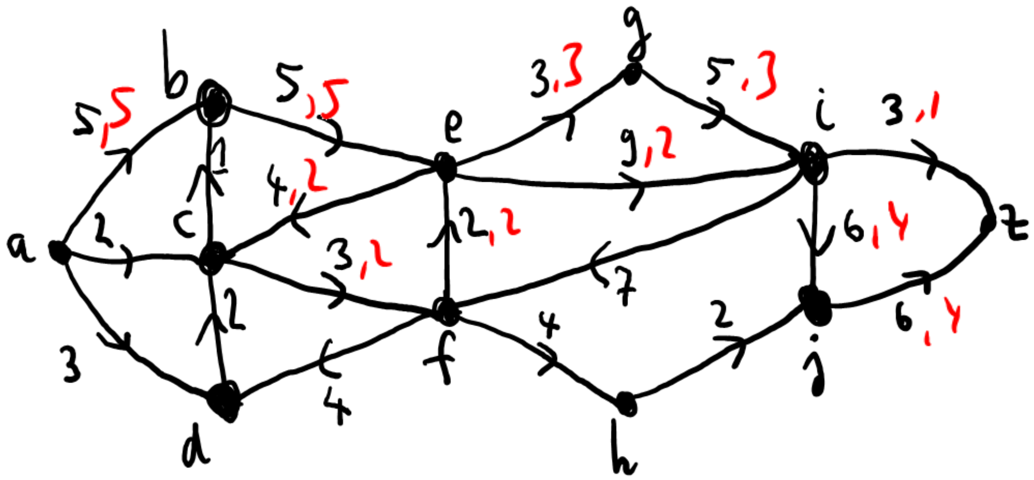
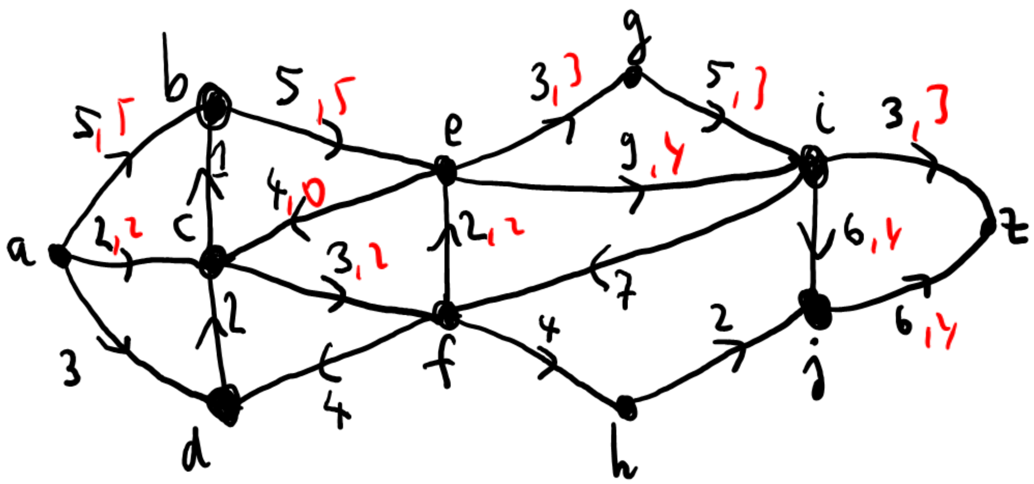


Draw after each iteration in the algorithm the improved flow RIGHT HERE into the network (next to the capacities). You must start with the above given flow. Indicate when you have found a maximal flow. (You may not need all diagrams.). **Solution:**





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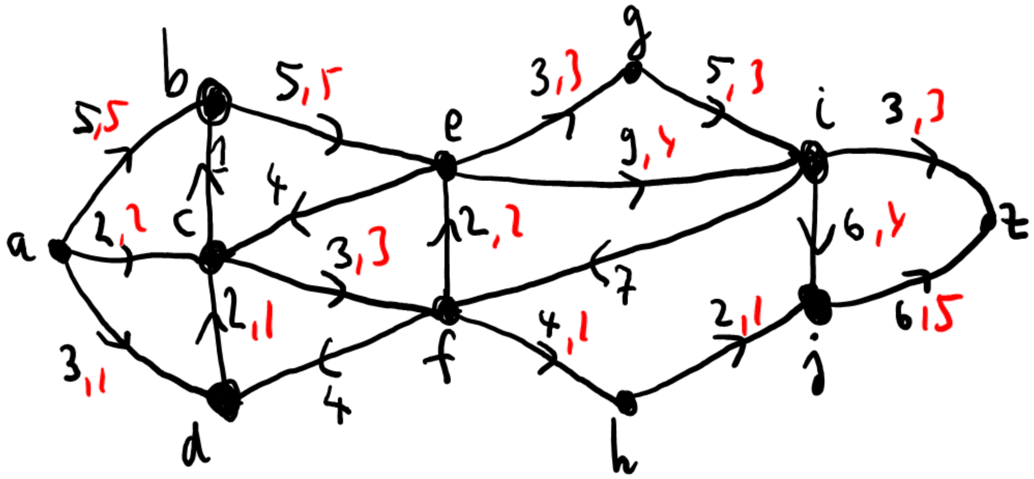

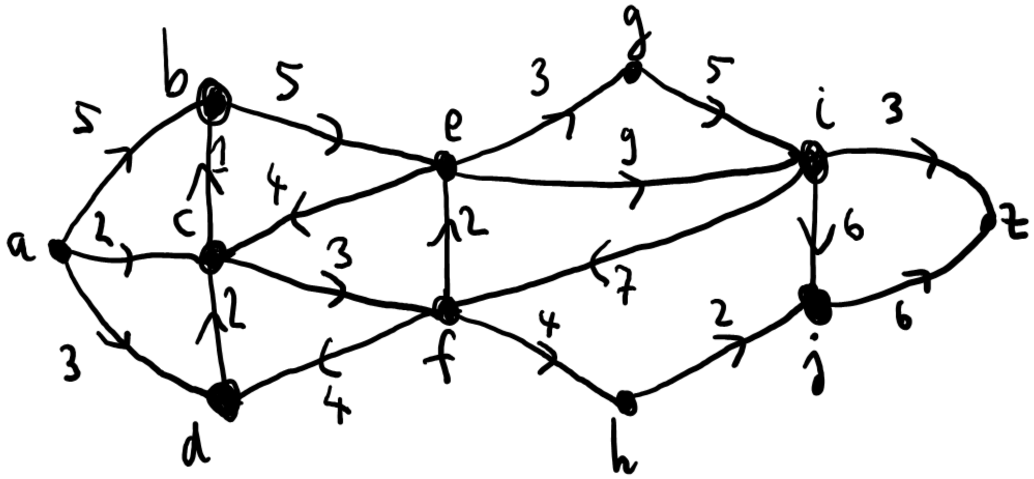

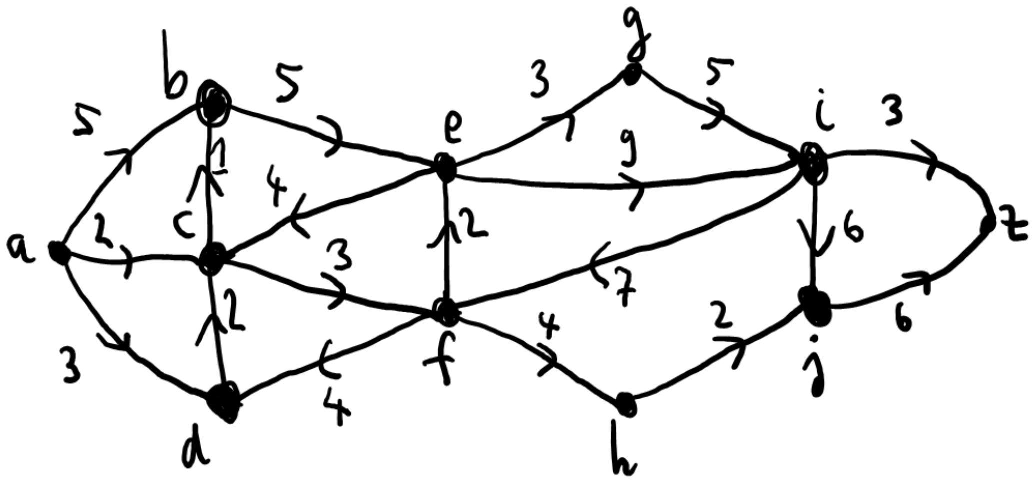

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Write here the flow value of the maximal flow you found:

Solution:

8

(b) (1 points) Give the cut with the minimal cut capacity RIGHT HERE that corresponds to the output of the algorithm above:

Solution:

$P = \{a, b, c, d\}$, and $\bar{P} = \{e, f, g, h, i, j, z\}$.

Write here the cut capacity of your cut:

Solution:

8