STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT7023
TENTAMEN
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First exam Stochastic Processes and Simulation II

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5 problems. Maximum of 60 points

In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

	A	B	C	D	E
Needed points	54	48	42	36	30

Partial answers (such as, not computing integrals) might be worth points, even if you cannot finish an answer! Answers “out of the blue” will not be rewarded.

Problem 1: Poisson Processes

a) Give the definition of a homogeneous Poisson Process. (4p)

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process with rate λ.

b) For $s > 0$, give the distribution of $N(s)$ conditioned on $N(1) = n$. (5p)

Hint: treat $s \in (0, 1)$ and $s > 1$ separately.

c) Let $X_1, X_2, \cdots$ be the points of the Poisson process $\{N(t), t \geq 0\}$.
Compute $\mathbb{E}\left(\sum_{i=1}^{N(1)} X_i\right)$. Assume $\sum_{i=1}^{0} X_i = 0$. (3p)

Problem 2: Renewal Theory

Assume busses arrive with independent and identically distributed interarrival times. The interarrival times have distribution function $F(t) = 1 - \frac{1}{t^3}$ for $t > 1$ and $F(t) = 0$ otherwise, i.e. $F(t) = (1 - \frac{1}{t^3})\mathbb{1}(t > 1)$.

a) Compute the expectation and variance of the interarrival times. (4p)

Passengers arrive according to a homogeneous Poisson proces with rate λ, independently of the arrival times of busses. A bus takes in all waiting passengers, after which new customers arrive to wait for the new bus.

b) What is the average number of customers waiting for the bus, if this average is taken over a very long time? (4p)

Hint: You can approach this problem to using an appropriate renewal reward process

c) What is the average time a customer have to wait for a bus to come? (4p)

Problem 3: Queueing Theory

Consider an $M/M/2$ queue in which customers arrive at rate λ, workloads are exponential with mean $1/\mu$ (i.e. customers which are served depart at rate μ), and there are two servers serving customers.

a) What relation should λ and μ satisfy in order for the number of customers not to grow beyond all bounds if time goes on? (3p)

Assume from now on that the condition from part a) is satisfied.

b) Provide the assymptotic distribution of the number of people in the system. (6p)

c) What is the average time (measured over a long period) that a customer stays in the system? (i.e. the time spend in the queue plus the time spend being served). (3p)

Problem 4: Brownian Motion and Stationary Processes

Let $\{B_1(t), t \geq 0\}$ is a Brownian motion with $B_1(0) = 1$, and variance parameter $\sigma_1^2 = 1/2$ and $\{B_2(t), t \geq 0\}$ is a Brownian motion independent of $\{B_1(t), t \geq 0\}$ with $B_2(0) = 0$, and variance parameter $\sigma_2^2 = 1/2$. Let

$$\{X(t), t \geq 0\} = \{B_1(t) - B_2(t), t \geq 0\} \text{ and } \{Y(t), t \geq 0\} = \{B_1(t) + B_2(t), t \geq 0\}$$

a) Show that $Cov(X(t), Y(t)) = 0$ and that both $X(t)$ and $Y(t)$ are normal distributed with mean 1 and variance t. (4p)

Using part a), it is possible to show that $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are independent Brownian motions with $X(0) = Y(0) = 1$ and variance parameter $\sigma^2 = 1$. You may use this without further proof in the following sub problems.

b) Let

$$T_0 = \min\{t \geq 0; B_1(t) = B_2(t)\} = \min\{t \geq 0; X(t) \leq 0\},$$

be the first time the Brownian motions $\{B_1(t), t \geq 0\}$ and $\{B_2(t), t \geq 0\}$ take the same value. Note that this time is distributed as the hitting time of 0 by the Brownian motion $\{X(t), t \geq 0\}$. Prove that T_0 has distribution function $2\mathbb{P}(X(t) \leq 0)$ and density function $\frac{1}{\sqrt{2t^3\pi}}e^{-1/2t}$. (5p)

Hint: You might consider the process defined through $\hat{X}(t) = 1 - X(t)$ and note that $T_0 = \min\{t \geq 0; \hat{X}(t) \geq 1\}$.

Let R_0 be the position where the Brownian motions $\{B_1(t), t \geq 0\}$ and $\{B_2(t), t \geq 0\}$ take the same value for the first time. Define $C_0 = 2R_0$. Note that $C_0 = Y(T_0)$, where $T_0 = \min\{t \geq 0, X(t) \leq 0\}$ has density function $\frac{1}{\sqrt{2t^3\pi}}e^{-1/2t}$ as in part b) and is independent of $\{Y(t), t \geq 0\}$.

c) Show that C_0 has density function: $\frac{1}{\pi}\frac{1}{1+(x-1)^2}$. (3p)

Note: Partial answers give points.

Problem 5: Simulation

Let X be a gamma distributed random variable with parameters n and λ, where n is a positive integer and λ a positive real number. This means $f_X(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}$. Let

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} f_X(x) dx$$

be the moment generating function of X.

a) Argue that X is distributed as $-\frac{1}{\lambda}\sum_{i=1}^n \log[U_i]$, where $U_1, U_2, \cdots, U_n$ are independent uniform random variables taking values on $[0, 1]$. (4p)

b) Let $t < \lambda$, Show that the "tilted" random variable X_t, with density function $f_{X,t}(x) = \frac{e^{tx} f_X(x)}{M_X(t)}$ is gamma distributed with parameters n and $\lambda - t$. (4p)

Suppose we want to estimate

$$\theta = \mathbb{P}(X > \alpha n/\lambda) = \mathbb{E}[\mathbb{1}(X > \alpha n/\lambda)],$$

where $\alpha > 1$. To do this we use importance sampling using the tilted density function $f_{X,t}(x) = \frac{e^{tx} f_X(x)}{M_X(t)}$.

c) Show that for all $t > 0$ we have: $\theta \leq M_X(t) e^{-t\alpha n/\lambda}$. (4p)