ALGEBRAIC GROUPS

WUSHI GOLDRING

- The course will be co-taught by Rikard Bögvad and myself.
- The following is an initial indication regarding the structure and contents of the course; changes and further details may be added later.
- Feel free to email me if you have any questions regarding the course.

1. Synopsis

It is not an exaggeration to say that algebraic groups play a key role in almost every area of mathematics. There has also been a deep interplay between algebraic groups and several areas in physics. Some key aspects of the theory of algebraic groups arose from physics. In turn, algebraic groups have had numerous applications to physics.

In many subjects, algebraic groups are increasingly appearing at the forefront, as in number theory, algebraic geometry, representation theory, Hodge theory, etc. In other areas algebraic groups remain hidden in the background, but even there one may argue that their importance will come to light in the future.

Besides their ubiquity, an additional attraction of algebraic groups is that one can *work with them in practice* without getting too entangled in the proofs of technical results about them.

The aim of this course will be to provide an appreciation for algebraic groups and connected structures such as root data and flag varieties by focusing on examples and applications, while keeping the prerequisites and technicalities to a minimum.

2. Prerequisites

- The course will be open to both Master's and PhD students.
- The only formal prerequisite is a first course in Abstract Algebra (MM5020 or equivalent).
- You will probably find it helpful if you have taken one of the courses Galois Theory, Topology or Commutative Algebra and Algebraic Geometry. However, none of these is logically required for Algebraic Groups.

3. Syllabus

3.1. **Overview and Examples.**

3.1.1. *Examples of classical algebraic groups and their Lie algebras.*

3.1.2. *Overview of varieties and definition of algebraic groups.*

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Date: June 19, 2017.

3.1.3. *Scope of the theory.*

- Different base field of interest
- Connection with Lie groups
- Connection with compact groups
- Motivational discussion about connections to other topics, as mentioned above.

3.2. **Structure theory and classification.**

3.2.1. *Root systems and root data; Weyl group.*

3.2.2. *Tori.*

3.2.3. *Root systems and root data for classical groups.*

3.2.4. *Discussion of the classification theorem.*

3.2.5. *The dual group.*

3.2.6. *Levi and Parabolic subgroups via root data.*

3.3. **Connection to geometry via flag varieties.**

3.3.1. *Examples of classical flag varieties.*

3.3.2. *Flag varieties via Borel and parabolic subgroups.*

3.3.3. *Flag varieties over finite fields: Counting points.*

3.4. **Finite-dimensional representations.**

3.4.1. *Examples of representations.*

- Tensor constructions: Symmetric and exterior powers; tensor product of representations
- Examples of geometric representations via the flag variety

3.4.2. *Theory of the highest weight.*

3.5. **Possible selection of more advanced topics.**

- Based on student interest and time constraints
- Most likely there will only be time to cover one or two such topics
- Spend one-two hours to show how algebraic groups connect to a more advanced topic which lies at the center of current research.
- Interested students can choose to pursue such a topic more in depth in a Master's thesis or future courses.

3.5.1. *Infinite-dimensional representations of real groups.*

3.5.2. *Complex Reflection Groups.*

3.5.3. *Schubert Varieties.*

3.5.4. *Connection with Hodge theory via period domains.*

4. Useful texts

4.1. **Knapp's book** [3]**.** Gives an elementary introduction to the structure theory and finite-dimensional representations of Lie groups. Even though it is written in the setting of Lie groups, it can be used to provide many explicit examples of algebraic groups and their representations. Chapter II gives a concrete introduction to complex simple Lie algebras, their root systems and classification.

4.2. **Springer's overview article** [4]**.** Gives a concise overview of the structure theory of algebraic groups, including the classification theorem in terms of root data. Also explains how the general theory specializes over particular fields, such as $\mathbf{R}, \mathbf{C}, \overline{\mathbf{F}}_p$.

4.3. **Jantzen's book** [2]**.** This book is very rich. It contains almost all of the topics mentioned above. The downside is that it assumes more algebraic geometry, which can make it difficult for a student to read on his/her own. On the other hand, many of the arguments can be translated by the instructor to a more elementary language. A strong point of the book is that it highlights the connection to geometry via flag varieties.

4.4. **Some other textbooks on algebraic groups.** Two other elementary introductions to algebraic groups are the books by Springer [5] and by Humphreys [1]. They can be used to supplement and complement the references above.

References

[1] J. Humphreys. *Linear algebraic groups.* Springer Verlag, 1975.

[2] J. Jantzen. *Representations of algebraic groups*, volume 107 of *Math. Surveys and Monographs.* American Mathematical Society, Providence, RI, 2nd edition, 2003.

[3] A. Knapp. *Lie groups Beyond an Introduction*, volume 140 of *Progress in Math.* Birkhauser, 1996.

[4] T. Springer. Reductive groups. In A. Borel and W. Casselman, editors, *Automorphic Forms, representations, and L-Functions*, volume 33 of *Proc. Symp. Pure Math.*, pages 3–28, Corvallis, OR, USA, July 11 - August 5 1979. Amer. Math. Soc.

[5] T. Springer. *Linear Algebraic Groups*, volume 9 of *Progress in Math.* Birkhauser, 2nd edition, 1998.