# HOMEWORK

WUSHI GOLDRING

- Due Thursday September 28 at 15:30 (either by email to me or hand-in to Samuel in class)
- There are 10 problems total, each worth 5 points.

## 1. Problems related to Lecture 1

In class we discussed Euclid's proof that there are infinitely many primes.

**Problem 1.1.** *(5 points) Modify Euclid's argument to show that there are infinitely many primes congruent to* 3 *modulo* 4*.*

In class we saw why a number which is congruent to 3 modulo 4 cannot be a sum of two squares.

**Problem 1.2.** *(5 points) Show that a number which is congruent to* 7 *modulo* 8 *cannot be a sum of* 3 *squares.*

**Problem 1.3.** *(5 points) Assume an integer $n$ is a sum of three squares: $n = x^2 + y^2 + z^2$ for some integers $x, y, z$. If $n$ is divisible by* 4*, show that also $n/4$ is a sum of* 3 *squares.* Hint: *Try to modify $x, y, z$ to three new numbers $x_1, y_1, z_1$ so that $x_1^2 + y_1^2 + z_1^2 = n/4$.*

In class we discussed some famous examples of Diophantine equations that have no rational solutions (like Fermat's Last Theorem) and some that have at most finitely many rational solutions (like Faltings' Theorem). For some Diophantine equations, it is much easier to show that there are no solutions by using congruences.

**Problem 1.4.** *(5 points) Show that the equation $x^2 + y^2 = 3z^2$ has no integer solutions.* Hint: *First argue why if there is a solution, then there is a solution $x_1^2 + y_1^2 = 3z_1^2$ where all of $x_1, y_1, z_1$ are relatively prime. Look at this equation modulo some well-chosen small number, show that there are no solutions modulo that number and deduce that there are no integer solutions.*

## 2. Problems related to Lecture 2

**Problem 2.1.** *(5 points) Find the quadratic residue classes modulo* 13*.*

**Problem 2.2.** *(5 points) Express $73 \cdot 101 \cdot 103^2$ as a sum of two squares.*

**Problem 2.3.** *(5 points) Factor* 1001 *and use Fermat's theorem to explain whether or not* 1001 *is a sum of two squares.*

## 3. Problems related to Lecture 3

**Problem 3.1.** *(5 points) Use quadratic reciprocity and the other properties of the Legendre symbol to compute the following three Legendre symbols:* $(7/71)$, $(63/79)$, $(61/107)$*.*

**Problem 3.2.** *(5 points) Use quadratic reciprocity to describe the odd primes $p$ such that* $(-3/p) = 1$

In Lecture 3, we will discuss Wilson's Theorem. It states that $(p-1)! \equiv -1 \pmod{p}$ for every prime $p$.

**Problem 3.3.** *(5 points) Use Wilson's Theorem to show that if $p$ is a prime, $p \equiv 1 \pmod{4}$, then $-1$ is a square mod $p$.*

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