Some Comments on Sets

A comment on a notation (a, b). It is unfortunate that this notation is firmly established in mathematics with two entirely different meanings. One is the ordered pair and the other is the set of all numbers x such that $a < x < b$. Most of time this conflict of notation will not cause difficulty because the meaning will be clear from the context. However there are situations where confusion is possible, In this case we use the symbol $a \times b$ for ordered pair.

Some words on image and preimage of function $f : A \to B$. If A_0 is a subset of A, B_0 is a subset of B.

Image of A_0: $f(A_0) = \{b : b = f(a) \text{ for at least one } a \in A_0\}$

Preimage of B_0: $f^{-1}(B_0) = \{a : f(a) \in B_0\}$

Note that

1. $f^{-1}(B_0)$ can be empty.
2. if f is bijective, we have two meanings for the notation $f^{-1}(B_0)$: (i) it is the premiere of B_0; (ii) it is the image of the function $f^{-1} : B \to A$. The two meanings give precisely the same subset of A.
3. The operation f^{-1} applied to subsets of B behaves nicely; it preserves inclusions, unions, interceptions, and differences of sets. We use this fact frequently. But the operation f applied to subsets of A preserves only inclusions and unions. More precisely, assuming $A_i \subset A$ and $B_i \subset B$ for $i = 0, 1$:
   (a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$
   (b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$
   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$
   (d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$
   (e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
   (f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$
   (g) $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$; the equality holds if f is injective
   (h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; the equality holds if f is injective
4. In general $f^{-1}(f(A_0)) = A_0$ and $f(f^{-1}(B_0)) = B_0$ are not true. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x^2 + 2$. Let $[a, b]$ denote the closed interval $a \leq x \leq b$. Then

   $$f^{-1}(f([0, 1])) = f^{-1}([2, 5]) = [-1, 1] \text{ and } f(f^{-1}(0, 5])) = f([-1, 1]) = [2, 5]$$

   But if $f : A \to B$ and if $A_0 \subset A$ and $B_0 \subset B$, then

   $$A_0 \subset f^{-1}(f(A_0))$$

   the equality holds if f is injective, and

   $$f(f^{-1}(B_0)) \subset B_0$$

   the equality holds if f is surjective.