Solutions to Kombinatorik 17/8 2010

1a) 3^{12}
b) $14!/(12!2!) = 91$
c) By looking at diagrams you see that there are the same number of partitions of 12 in at most 3 parts as there are partitions of 12 in parts ≤ 3. The generating function for this is $((1-x)(1-x^2)(1-x^3))^{-1}$. The coefficient of x^{12} is 19.
d) Inclusion-exclusion gives $3^{12} - 3 \cdot 2^{12} + 3 \cdot 1^{12}$. (The total number is 3^{12}, first box empty gives 2^{10} and the same for the other boxes, box 1 and box 2 empty gives 1 and the same for two other empty boxes.)
e) Place first one boll in each box. Then you should place 9 bolls in 3 boxes without restriction, so there are $11!/(9!2!) = 55$ ways.
f) There is one partition in one part, and 6 partitions in two parts (1+11, 2+10, 3+9, 4+8, 5+7, 6+6). Thus there are 19-1-6=12.
g) See c).
h) Generating function $1/(1-x^2)1/(1-x^4)\cdots 1/(1-x^{12})$. The coeficcient of x^{12} is 11.
i) Generating function $1/(1-x)1/(1-x^3)\cdots 1/(1-x^{11})$. The coeficcient of x^{12} is 15.

2a) There are 4 A's, 3 T's, and one R, so there are $8!/(4!3!1!) = 280$.
b) By choosing 7 letters and putting the remaining last, you see that there are the same number of words with 7 letters.

3. Let $o(n)$ be the number of strings of length n with an odd number of even numbers, and let $e(n)$ be the number of strings with an even number of even elements. Then $o(n) + e(n) = 3^n$. We have

$$o(n) = o(n-1) + 2e(n-1) = 2 \cdot 3^{n-1} - o(n-1).$$

$o(n) + o(n-1) = 3^{n-1}$ and $o(1) = 2$ gives $o(n) = (3^n - (-1)^n)/2$.

4a) Yes, e.g. (45) – (12) – (34) – (15) – (23) – (14) – (25) – (13) – (24) – (35).
b) G is 3-regular (all vertices have degree 3), so there is no Euler circuit. (All degrees are even if there is an Euler circuit.)
c) No, there are odd cycles, e.g. (12) – (34) – (15) – (23) – (14) – (25) – (13) – (24) – (35) – (12).
d) Yes, e.g. (12) – (13) – (14) – (15) – (25) – (24) – (23) – (34) – (35) – (45).
e) $\bar{G}$ is 6-regular, so there are Euler circuits.
f) No, there are odd cycles, e.g. (12) – (13) – (23) – (24) – (14) – (12).

5a) E.g. (dg),(dh),(bd),(eh),(fi),(ce),(ij),(cf),(ab).
b) a-b-d-h-j
c) The maximal flow is 17. E.g. ab: 6, ac: 7, ae: 4, bd: 3, be: 3, ce: 2, cf: 5, dg: 2, dh: 1, eg: 4, eh: 3, ei: 2, fh: 2, fi: 3, gj: 6, hj: 6, ij: 5

Minimal cut: $\{a, b, c, d, e, f, h, i\} \cup \{g, j\}$.