Krull dimension

Symbolic Power: Let P be a prime ideal in ring R, and for $n \in \mathbb{N}$ the following ideal is called the n-th symbolic power of P.

$$P^{(n)} = \{a \in R \mid ab \in P^n \text{ for some } b \in R \setminus P\}$$

Lemma:

(a) $P^n \subseteq P^{(n)} \subseteq P$. (obvious)

(b) $P^{(n)}$ is P-primary.

(c) $P^{(n)} R_P = P^n R_P$.

Proof: (b): First see that $P = \sqrt{P^n} \subseteq \sqrt{P^{(n)}} \subseteq P = \sqrt{P} \longrightarrow \sqrt{P^{(n)}} = P$.

(it is enough to check this)

Now, take $ab \in P^{(n)}$ s.t $b \notin P$ we know that by Def above

why?

there exists $x \in R \setminus P$ s.t $abx \in P^n \subseteq P^{(n)}$. Since $b \notin P \rightarrow bx \notin P$

therefore $a \in P^{(n)}$.

(c) Since $P^{(n)} \supset P^n$ hence $P^{(n)}R_p \supseteq P^n R_p$. For the other direction take $\frac{b}{s} \in P^{(n)}R_p$ or $bc \in P^n$ for some $c, s \in R \setminus P$.

Thus $\frac{b}{s} = \frac{bc}{sc} \in P^n R_p$. ✓

Def: "Krull dimension": Let A be a ring then

$$\dim A = \sup\{n \in \mathbb{Z}^{\geq 0} \mid P_0 \subsetneq \cdots \subsetneq P_n \ ; \ P_i \in \operatorname{Spec} A\}$$

Def: If $P \in \operatorname{Spec} A$ then height of P is $\operatorname{ht}(P) = \sup\{n \mid P_0 \subsetneq \cdots \subsetneq P_n = P\}$

Example: $\dim \mathbb{Z} = 1$, $\dim k[x] = 1$ $\dim k = 0$

Krull Principal ideal theorem: Let R be Noetherian ring and let $a \in R$. Then the height of a minimal prime ideal P of $\langle a \rangle$ is atmost one.

Sketch of Proof: Consider a chain of Prime ideals $q' \subset q \subsetneq P$ in R.

we want to show that $q = q'$. WLOG, we can assume $q' = 0$ and A is local at P. This can be done by first taking $A := \frac{A}{q}$ and localizing at P. Now we want to show that $q = 0$!!!

Fact 1) $q^{(n)} = q^{(n+1)} + P q^{(n)}$. (Exercise)

See that $\frac{q^{(n)}}{q^{(n+1)}} = \frac{P q^{(n)}}{q^{(n+1)}}$ by Nakayama Lemma $\frac{q^{(n)}}{q^{(n+1)}} = 0$.

Now, $q^{(n)} = q^{(n+1)}$ implies $q^{(n)} R_q = q^{(n+1)} R_q$ and by Privious lemma

$q^n R_q = q^{n+1} R_q = (q R_q) q^n R_q$ again use Nakayama lemma

$q^n R_q = 0$. But R is an integral domain hence $q = 0$. ✓

Corollary: Let A is Noetherian and $P \subsetneq P' \subsetneq q$ then there infinitly many Prime ideals between P and q.

Proof: Assume that there are only finitely man P_i $1 \le i \le n$ in $\mathrm{Spec}(A)$ with $P \subsetneq P_i \subsetneq q$. If $q \subseteq \bigcup_{i=1}^{n} P_i$ then $q \subseteq P_i$ $\divideontimes$.

So there exists $x \in q \setminus \bigcup_{i=1}^{n} P_i$. Now, take $\bar{0} \subsetneq \frac{P'}{P} \subsetneq \frac{q}{P}$ in $\frac{A}{P}$.

$ht\left(\frac{q}{P}\right) \geq 2$ and $q \in \mathrm{Min}(x + P)$ $\divideontimes$

Exercise: If A is a Noetherian ring and x is not unit nor a zero divizor then every minimal prime ideal (x) has height 1.

Exercise: [Generalize krull Principal Thm] : Let A be a Noetherian ring and $P \in \mathrm{Min}(\langle a_1, \dots, a_r \rangle)$ with $\langle a_1, \dots, a_r \rangle \neq A$ then $ht\, P \leq r$.

Exercise: If A Noetherian and $P \in \mathrm{Spec}\, A$, then

$$\dim \frac{A}{P} + \operatorname{ht} P \leq \dim A$$