Notes on Predicate Logic

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Peter LeFanu Lumsdaine

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Note: the organisation of these notes is based on that of Carlström, *Logic*, Chapters 9–14.

These notes are incomplete and in progress, and will be expanded with exercises, more material, etc. as the course continues.

1 Syntax

1.1 Arity types

Definition 1.1. An *arity type* is a pair $\langle \vec{a}; \vec{r} \rangle$ of finite sequences of natural numbers.

An arity type should be read as setting up the operations and atomic predicates for a specific intended application of predicate logic.

The first sequence $\vec{a} = \langle a_1, a_2, \ldots, a_n$ specifies the arities of the operation symbols; so, for instance, $a_1 = 2$ would mean that f_1 represents a binary operation. Similarly, $a_2 = 4$ will make f_2 represent a function of four variables. A 0-ary (or *nullary*) function is just a constant term.

Similarly, the second sequence represents the arites of the atomic predicates.

1.2 Terms

We take variables as indexed by natural numbers.

Definition 1.2. Given some arity type $\langle \vec{a}; \vec{r} \rangle$, we define for each finite $S \subseteq \mathbb{N}$ the set $\mathrm{Term}(S)$, of *terms with variables indexed by S*, as inductively generated by the following rules:

$$\frac{i \in S}{x_i \in \mathrm{Term}(S)}$$

$$\frac{t_1 \in \mathrm{Term}(S) \qquad \cdots \qquad t_{a_i} \in \mathrm{Term}(S)}{f_i(t_1, \ldots, t_{a_i}) \in \mathrm{Term}(S)}$$

If we need to emphasise the dependence on the arity type, we will write $\mathrm{Term}_{\langle \vec{a};\vec{r}\rangle}(S)$. Usually, though, the arity type will be fixed and clear from context, so we will just write $\mathrm{Term}(S)$.

Often we will write e.g. $\mathrm{Term}[x_1, x_4, x_5]$ instead of $\mathrm{Term}(\{1,4,5\})$, for readability.

Note that a term $t \in \mathrm{Term}(S)$ does not need to mention all the variables in S. For example, $x_5 \in \mathrm{Term}[x_1, x_4, x_5]$ (by the first rule of the inductive definition), and if f_i is a nullary function symbol, then $f_i \in \mathrm{Term}(S)$ for any S.

Compare how, in real analysis or algebra, $y = 5$ is a perfectly good definition of y as a function of x, or of x and z, or of any other set of variables; it just happens to be a constant function. Our semantics will reflect this: a term $t \in \mathrm{Term}(S)$ will be interpreted as a function depending on variables indexed by S.

1.3 Formulas

To come.

1.4 Substitution

To come.

2 Semantics

2.1 Preliminaries

To come.

2.2 Structures

As usual, we assume an arity type $\langle \vec{a}, \vec{r}\rangle$ is given.

Definition 2.1. A *structure* or *interpretation* $\mathcal{A}$, for the arity type $\langle \vec{a}, \vec{r}\rangle$, consists of:

1. a set $|\mathcal{A}|$, the *domain* or *underlying set* of $\mathcal{A}$;
2. for each i, an a_i-ary operation $f_i^{\mathcal{A}}$ on $|\mathcal{A}|$, i.e. a function
$$f_i^{\mathcal{A}} : |\mathcal{A}|^{a_i} \to |\mathcal{A}|;$$
3. for each i, an r_i-ary predicate $P_i^{\mathcal{A}}$ on $|\mathcal{A}|$, i.e. a subset
$$P_i^{\mathcal{A}} \subseteq |\mathcal{A}|^{r_i}.$$

2.3 Interpretation of terms, formulae

To come.

2.4 Notions of logical consequence, etc.

To come.

3 Natural deduction

To come.