A Proof of Pick's Theorem

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Theorem *(Pick's Theorem).* Let P be a simple polygon in $\mathbb{R}^2$ such that all its vertices have integer coordinates, i.e., the vertex set of P is contained in the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let I and B denote the number of lattice points in the interior of and on the boundary of P respectively. Then the area A of P is given by

$$A = I + B/2 - 1.$$

Proof. This proof is based on the arguments given by Davis [1]. Throughout this proof, all considered polygons will have their vertices on lattice points.

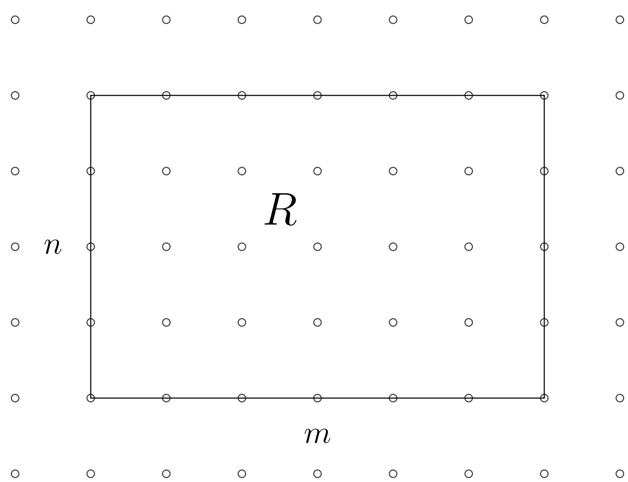

Figure 1: An $m \times n$ rectangle

Consider an $m \times n$ lattice aligned rectangle R (i.e., a rectangle with its edges parallel to the axes in our coordinate system). See Figure 1. Since m and n are the side lengths, of course $m, n \in \mathbb{Z}$. Let us count the number of lattice points I_R in the interior of R. It is easily seen that

$$I_R = (m-1)(n-1) = mn - m - n + 1.$$

Concerning the number of lattice points B_R on the boundary of R, it is equally straightforward to see that

$$B_R = 2m + 2n.$$

Consequently, we immediately find that Pick's theorem holds for any lattice-aligned rectangle R with area A_R, since

$$A_R = mn = mn + m + n - 1 + \frac{2m + 2n}{2} + 1 = I_R + B_R/2 - 1$$

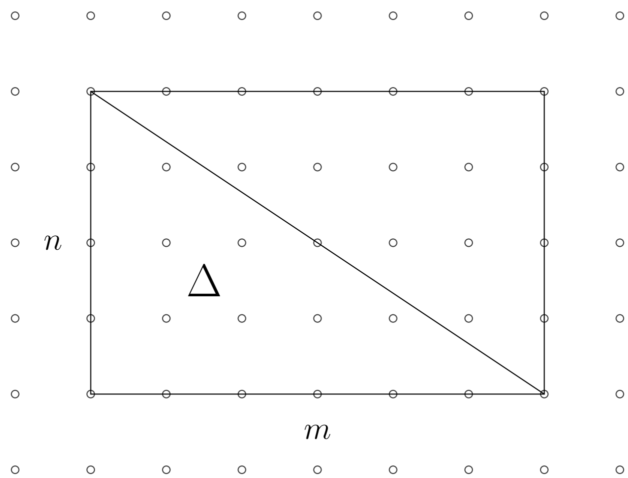

Figure 2: A right triangle Δ inscribed in an $m \times n$ rectangle

After this satisfying success, let us continue with considering a lattice aligned (two sides parallel to axes) right triangle Δ with non-hypothenuse sides m and n. This triangle can then be inscribed in a lattice aligned rectangle R such that they share two sides and the hypothenuse of Δ forms one of the diagonals of R. See Figure 2. The only non-triviality is the number of lattice points on the hypothenuse, so let us suppose that there are k such points (not counting corners of Δ). Then the number of lattice points B_Δ on the boundary of Δ is given by

$$B_\Delta = m + n + 1 + k$$

The number of lattice points I_Δ in the interior is given by half of the interior points of the rectangle after subtracting the ones on the hypothenuse, that is,

$$I_\Delta = \frac{(m-1)(n-1) - k}{2}$$

Let us try the formula:

$$I_\Delta + B_\Delta/2 - 1 = \frac{(m-1)(n-1) - k + m + n + 1 + k}{2} - 1 = \frac{mn}{2}$$

Luckily this expression happens to equal the area of Δ. Consequently, Pick's theorem also works for any lattice aligned right triangle Δ.

We will continue by proving Pick's theorem for any triangle (of course still with its vertices on lattice points). Let Δ be such a triangle. There are two cases to consider: *no side of* Δ *is parallel to an axis* and *one side of* Δ *is parallel to an axis*. We can safely reassure any

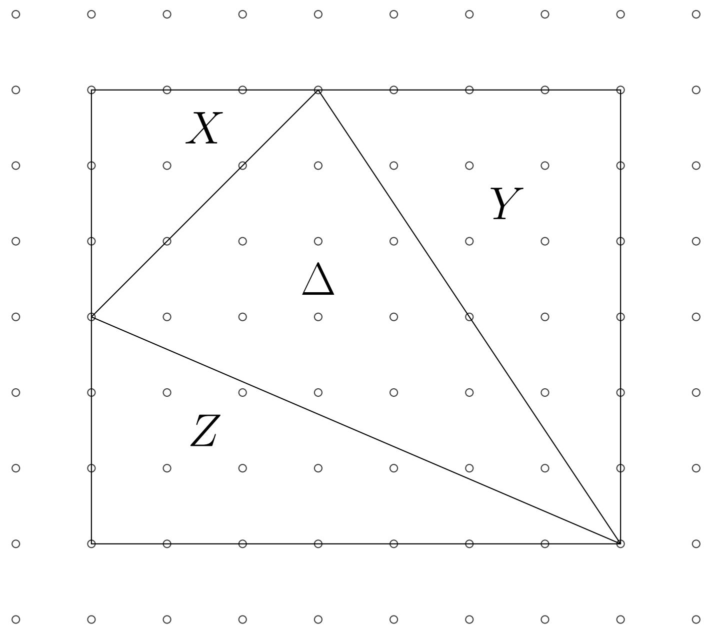

Figure 3: A more general triangle inscribed in an $m \times n$ rectangle

reader thinking that a case is missing (two of the sides parallel to axes), by noting that this has already been covered since it is a right triangle.

Let us start with the first case. Any such triangle Δ can be rotated and/or flipped so that we end up in the situation depicted in Figure 3. We know that Pick's theorem works for the triangles X, Y and Z and the enclosing rectangle R. Thus we obtain the following equations.

$$A_X = I_X + B_X/2 - 1$$
$$A_Y = I_Y + B_Y/2 - 1$$
$$A_Z = I_Z + B_Z/2 - 1$$
$$A_R = I_R + B_R/2 - 1$$

Furthermore, we can describe relevant quantities of Δ in the following manner.

$$A_\Delta = A_R - (A_X + A_Y + A_Z)$$
$$B_\Delta = B_X + B_Y + B_Z - B_R$$
$$I_\Delta = I_R - (I_X + I_Y + I_Z + B_\Delta) + 3$$

The "+3" is a compensation for the substraction of the whole boundary of Δ including its vertices. Combining these equations we obtain (after some algebraic manipulations)

$$A_\Delta = A_R - (A_X + A_Y + A_Z) =$$

$$= I_R + B_R/2 - 1 - (I_X + B_X/2 - 1 + I_Y + B_Y/2 - 1 + I_Z + B_Z/2 - 1) =$$

$$= (I_R - I_X - I_Y - I_Z - B_\Delta + 3) + B_\Delta - 3 + B_R/2 - (B_X/2 + B_Y/2 + B_Z/2) + 2 =$$
$$= I_\Delta + B_X + B_Y + B_Z - B_R + B_R/2 - (B_X/2 + B_Y/2 + B_Z/2) - 1 =$$
$$= I_\Delta + \frac{B_X + B_Y + B_Z - B_R}{2} - 1 = I_\Delta + \frac{B_\Delta}{2} - 1$$

which shows that Pick's theorem holds true for any triangle Δ described above.

Concerning the other case where we let Δ be a triangle that shares one side with its enclosing rectangle, we need only consider the construction as in Figure 4. By changing Z into a rectangle below Δ we can in fact use the same proof as above also for this case.

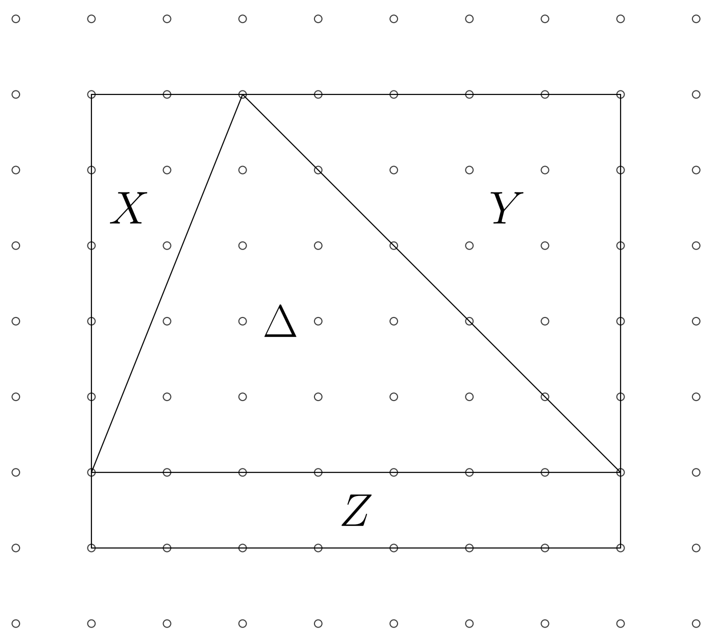

Figure 4: A triangle with one side parallel to an axis

To this point we have proven that Pick's theorem holds for all triangles and we will proceed to prove it for any polygon by induction. Suppose that the theorem holds true for any simple polygon with less than n vertices. We will show that this implies that for any simple polygon P with n vertices we have

$$A_P = I_P + B_P/2 - 1$$

Consider a simple polygon P with n vertices. We will use the known fact that P can be triangulated. This implies that we can find a triangle T and remove it from P obtaining another polygon P' with $n-1$ vertices. See Figure 5. Let k be the number of vertices on the boundary between T and P'. Then one realizes that

$$I_P = I_{P'} + I_T + k - 2$$

where the "-2" stems from two of the k points that lie on the boundary of P. Furthermore

$$B_P = B_{P'} + B_T - 2k + 2.$$

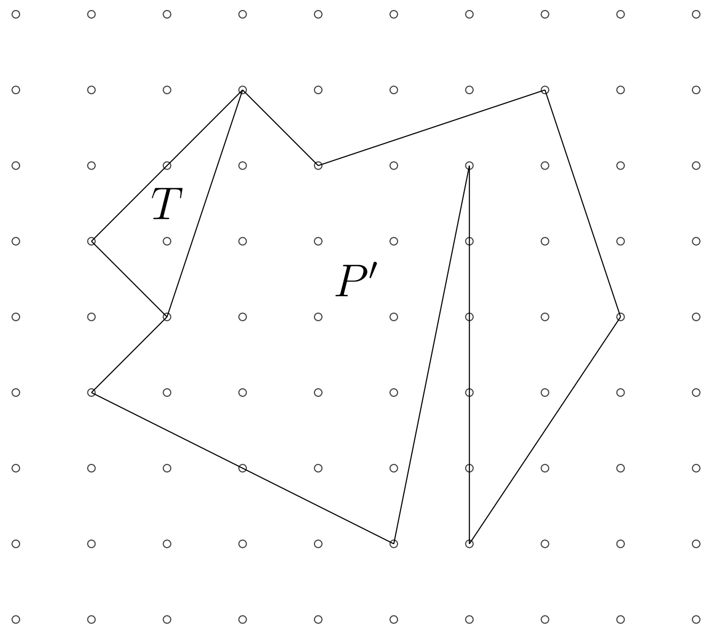

Figure 5: A polygon P constructed from a smaller polygon P' and a triangle T

To finish this off, let us consider

$$A_P = A'_P + A_T = I_{P'} + B_{P'}/2 - 1 + I_T + B_T/2 - 1 =$$
$$= (I_{P'} + I_T - 2 + k) - k + \frac{B_{P'} + B_T}{2} + 1 - 1 =$$
$$= I_P + \frac{B_{P'} + B_T - 2k + 2}{2} - 1 = I_P + B_P/2 - 1$$

and thus Pick's theorem holds for P. Since we began with proving the theorem for the base case $k = 3$ (corresponding to triangles) the proof is complete.

$\square$

References

[1] Davis, T. *Pick's Theorem* (Oct 27, 2003).
Retrieved from http://www.geometer.org/mathcircles/pick.pdf