STOCKHOLMS UNIVERSITET	MT5012
MATEMATISKA INSTITUTIONEN	TENTAMEN
Avd. Matematisk statistik	Aug 15, 2023

Second exam: Stochastic Processes and Simulation II

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Grading: The exam consists of five exercises, each divided into three questions. Each question is worth a maximum of 4 points, for a total of 12 points per exercise. Exercise 4 contains a bonus question which is worth 2 extra points, for a total of 14 points. The maximum score for the exam is then 62 points.
In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

Grade	A	B	C	D	E
Needed points	54	48	42	36	30

Partial answers might be worth points, while answers "out of the blue" will not be rewarded. The questions differ in difficulty.

Exercise 1: Poisson processes

(i) Give the definition of a compound Poisson process $\{X(t), t \geqslant 0\}$ with parameter λ and distribution F. Write an expression for its mean $\mathbb{E}[X(t)]$ and its variance $\mathrm{Var}(X(t))$.

One of the many consequences of the climate crisis is the increasing number of wildfires occurring over the summer in Southern European countries such as Portugal, Spain, Italy and Greece. As global temperatures rise, the hot and dry conditions help these fires catch and spread, and it is expected that their size, frequency and severity will increase in the coming years, with extreme wildfires devastating communities and ecosystems.
Assume that in Spain wildfires occur over the summer according to a Poisson process with a rate of $\lambda_S = 2$ per week. Suppose also that each time a wildfire occurs, nearby villages are evacuated and people are relocated, and that the number of relocated people N_S has distribution F_S given by

$$P(N_S = 100) = 1/4, \qquad P(N_S = 300) = 1/2, \qquad P(N_S = 500) = 1/4,$$

independently for each wildfire.

(ii) Let $S(t)$ be the total number of relocated people due to wildfires in Spain over the summer. What type of process is $\{S(t), t \geqslant 0\}$? Calculate the expected value and variance in a summer month, i.e., calculate $\mathbb{E}[S(4)]$ and $\mathrm{Var}(S(4))$, where we approximate 1 month $\approx$ 4 weeks.

(iii) Assume that in Portugal wildfires occur over the summer according to a Poisson process with rate λ_P per week and that the number of relocated people N_P has distribution F_P, independently for each wildfire. Assume also that wildfires in Portugal occur independently from wildfires in Spain. Let $P(t)$ be the total number of relocated people due to wildfires in Portugal over the summer. What type of process is $\{S(t) + P(t), t \geqslant 0\}$? What is its parameter and its distribution, in terms of $\lambda_S, \lambda_P, F_S, F_P$?

Exercise 2: Renewal theory

(i) Let $\{N(t), t \geqslant 0\}$ be a renewal process with i.i.d. interarrival times X_n, $n \geqslant 1$. Let $\mu = \mathbb{E}[X_n]$ and let $m(t) = \mathbb{E}[N(t)]$ be the renewal function. The elementary renewal theorem states that $\frac{m(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$. Prove the lower bound, i.e., that $\lim_{t\to\infty} \frac{m(t)}{t} \geqslant \frac{1}{\mu}$.

(ii) Let $\{N(t), t \geqslant 0\}$ be as above and consider a renewal reward process $\{R(t) = \sum_{n=1}^{N(t)} R_n, t \geqslant 0\}$ where R_n, $n \geqslant 1$ are i.i.d. and represent the rewards earned each time a renewal occurs. State and prove the reward theorem for $\frac{R(t)}{t}$.

Suppose that each time a wildfire occurs a relocation center is installed nearby in order to help people evacuate and move to the nearest city. Assume that people arrive at the relocation center according to a Poisson process with a rate of 10 per hour, and that, as soon as there are 30 people, a bus picks them all up and departs. The bus service association incurs a cost at a rate of $4k$ euros per unit time whenever there are k people waiting at the relocation center.

(iii) Describe the problem in terms of a renewal reward process. What is the expected length of a cycle? What is the expected cost in a cycle? Use the renewal reward theorem to compute the long-run average cost.

Exercise 3: Queueing theory

Assume that when people arrive at the relocation center they first have to go to a registration room to identify themselves. Assume they arrive at a Poisson rate λ independently of each other and they form a single queue for two registration desks: one is always active, while the other one is active if and only if there are at least 3 people in the registration room. Assume that the time it takes at each desk is exponentially distributed with mean $1/\mu$, independently of everything else.

(i) Specify what type of queueing model best describes the registration process. What condition must λ and μ satisfy in order for the number of people not to grow beyond all bounds?

(ii) Write down the balance equations for the above queueing system.

(iii) Let μ_R be the average number of people in the registration room. What is the asymptotic average time that a person spends in the registration room? What is the asymptotic average time that a person spends in queue in the registration room?

Exercise 4: Simulation

(i) Assume that wildfires in Spain occur over the summer according to a Poisson process $\{W_S(t), t \geqslant 0\}$ with rate $\lambda_S = 2$. Describe how we can simulate this process by simulating only standard uniform random variables.

Assume that in Italy wildfires occur over the summer according to a nonhomogeneous Poisson process $\{W_I(t), t \geqslant 0\}$ with intensity function $\lambda_I(t) = \frac{1}{t+3}$.

(ii) In Italy, given that a wildfire occurs at time x, compute the density function $f_x(t)$ of the time at which the next wildfire occurs.

(iii) Describe how we can simulate the occurrence of wildfires over the summer in Italy by simulating only standard uniform random variables.

(iv) Bonus (2 points): how can we simulate the time of the first wildfire that occurs either in Spain or Italy?

Exercise 5: Brownian motion

(i) Give the definition of a Brownian motion $\{X(t), t \geqslant 0\}$ with drift coefficient μ and variance parameter σ^2. Show that $\frac{X(t)}{t} \to \mu$ almost surely as $t \to \infty$.

Wildfires in Southern Europe are expected to increase in the coming years, due to global warming and temperature rise. Indeed, we know that in 2023 the global temperature has increased by 1.1 degrees Celsius compared with pre-industrial levels, and it is predicted to keep increasing in the next decades.

(ii) Assume that the global temperature evolves according to a Brownian motion with drift and that exactly in 2035 it will be 1.5 degrees Celsius higher compared with pre-industrial levels. What is the value of the drift coefficient μ in the unit Celsius/years? In which year it is expected to reach the 2 degrees Celsius threshold?

Assume that we will be able to find solutions to the climate crisis and manage to half the effect of the drift in the year 2035 and to completely cancel it in the year 2041, so that the global temperature will evolve according to a Brownian motion with drift μ until 2035, then with drift $\mu/2$ until 2041, and then without drift.

(iii) If in 2041 the global temperature will be exactly at its mean value and if $\sigma = 1$, what is the probability that it will not reach the 2 degrees Celsius threshold by 2050?