STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5012
TENTAMEN
August 13, 2021

Second exam Stochastic Processes and Simulation II

August 13, 2021 kl. 9–15

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Preliminaries

This exam is a home-exam. The candidate may use books, mathematical software, material available on the internet etc. If you use results from the 11th edition of "Introduction to Probability Models" by Sheldon Ross, you do not have to copy the argument but a proper reference (including page number) is sufficient. Otherwise proper referencing is also needed, but arguments need to be copied or paraphrased.

The exam should be handed in August 13 no later than 15:00 (deadline). This should be done using the function given on the course website (near bottom of website). If problems arise, send solution by email to ptrapman@math.su.se.

The solutions to the exam should be submitted on the course homepage as a single PDF file. There are no restrictions regarding what your PDF should contain. For example, the PDF may be based on a Word document, a Latex document, or scanned nicely handwritten solutions. If you plan on "scanning" handwritten solutions using your mobile phone, I suggest downloading and using a "scanning app". If you scan and thereby obtain several PDF files, then there are many programs that can be used to merge several PDF files into one PDF file.

The home exam will follow the same style as old exams. Hence, your solution should be of the same type as for usual exams (i.e. not of "thesis type").

If something is unclear or if you experience problems during the exam, please notify me as soon as possible by sending an e-mail to ptrapman@math.su.se

If I need to get in touch with you during the exam I will use the news forum on the course home page, so please check this regularly.

If you want to ask questions, that are hard to formulate by email, concerning the exam you may send an e-mail to ptrapman@math.su.se with the subject "Exam MT5012" together with a Zoom meeting ID. I will contact you as soon as possible.

Grading

The exam consists of 5 problems each divided in subproblems. Maximum of 60 points.

In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

	A	B	C	D	E
Needed points	50	45	40	35	30

Partial answers (such as not evaluating integrals or sums) might be worth points, sometimes even a maximal score! Answers "out of the blue" will not be rewarded.

The problems differ in difficulty. Do not be alarmed if you think the question is too easy, and only require a one line answer. It might be just that.

Statement by student

Write out in full on the exam and sign that you promise that you

1. have written the exam without the assistance of any other person,

2. have not discussed the exam with anyone while writing it (apart from possibly the lecturer).

Problem 1: Poisson Processes

Adam is sitting at the roadside watching cars passing by and counting how many people are in each car. Cars are passing by according to a homogeneous Poisson process $\{N(t); t \geq 0\}$ with intensity $\lambda > 0$. Independently for each car, the number of people in the car (including the driver) is one with probability p_1, two with probabilitly p_2 and three with probability p_3, where p_1, p_2 and p_3 are in the interval $(0, 1)$ and $p_1 + p_2 + p_3 = 1$.

a) Provide the density function of the time at which Adam sees the second car. (4pt)

b) Let X be the number of cars with 1 person in them that has passed at time 10, and Y be the number of cars with 2 people in them that has passed at time 10. Compute $Cov(X, Y)$, which is the covariance of X and Y. (4pt)

c) Let $\{C(t); t \geq 0\}$ be the process describing the total number of people in cars that have passed Adam. Let T be a fixed strictly positve time. Compute $\mathbb{E}[C(T)]$ and $Var[C(T)]$. (4pt)

Problem 2: Renewal Theory

In this exercise α, β, c, C_L and T are strictly positive constants, while K is a strictly positive integer. We consider the following model for the spread of an epidemic in a population.

- If there are no infectious people in the population the time until a peson is infected from outside the population is exponentially distributed with expectation $1/\alpha$.
- If there are $k \in \{1, 2, \cdots, K-1\}$ infectious people in the population, a new person gets infected after an exponentially distributed time with expectation $1/(k \times \beta)$.
- After the K-th infection (say at time τ_K), new people get infected according to an inhomogeneous Poisson process with intensity $Ke^{\beta(t-\tau_K)}$ on the interval $(\tau_K, \tau_K + T)$.
- At time $\tau_K + T$ the number of infectious people becomes 0 again through a very severe and immediate effective lockdown.
- The exponential distributions and the Poisson process are all independent on each other.
- An infected person stays infectious from infection until the first lockdown after infection.

In the questions below, some ugly sums and integrals might appear. You do not have to transform them to more appealing expressions.

a) What is the expected time between two subsequent lockdowns? (3pt)

b) What is the expected number of people infected between two subsequent lockdowns? (3pt)

Assume that the cost per infectious person per time unit for society is c and the cost of a lockdown for society is C_L.

c) What is the (almost sure) long-run average cost per time unit of this epidemic for this population? (6pt)

Hint 1: This is possibly the most involved question of the exam and not evaluating integrals will not cost many points in this question.

Hint 2: Find the total expected cost of the first K people infected after a lockdown, and the expected cost per person infected in the last T time units before lockdown.

Problem 3: Queueing Theory

Consider the following queueing system for a shop.

Let λ and μ be strictly positive constants and $c_0, c_1, c_2, \cdots$ be a decreasing sequence of real numbers in $[0, 1]$.

There is one server who works according to first come, first serve discipline. While waiting for service and while being in service every customer that enters is in the shop. He or she is in service for an exponentially distributed time with expectation $1/\mu$. Service times are independent. Customers come to the shop according to a homogeneous Poisson process with intensity λ. This Poisson process is independent of the service times. When a customer arrives at the shop he looks how many people are in. For $k \in \{0, 1, \cdots\}$, if there are already k customers in the shop, the new arrived customer enters with probability c_k and with probability $1 - c_k$ he or she leaves without entering. The decision on entering only depends on the number of customers already in the shop and not on other aspects of the process.

a) What additional condition should the sequence $c_0, c_1, c_2, \cdots$ satisfy in order for the number of customers in the shop not to go to infinity as time goes to infinity. (2pt)

Set $c_k = 1/(k + 1)$.

b) Provide balance equations for the stationary distribution of the number of customers in the system and show that the stationary distribution of the number of customers in the system has a Poisson distribution with expectation λ/μ. (5pt)

c) What is the long run fraction of customers that enter the shop among the customers that arrive at the shop. (5pt)

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t); t \geq 0\}$ is a standard Brownian motion. (So, $B(0) = 0$ and $Var(B(1)) = 1$.) Let a be a stricly positive constant

Let $\{X(t); t \geq 0\}$ be a regenerative process, which is defined as a standard Brownian motion until the process hits a barrier at level a, at which time a regenation takes place. That is, for all $k \in \{1, 2, \cdots\}$, as soon as the Brownian motion hits level a for the k-th time (say at time τ_k) then, without delay, the process is set back to 0 and thus $X(\tau_k) = 0$ for all $k \in \{1, 2, \cdots\}$.

Let $\{K(t); t \geq 0\}$ be the renewal process describing the number of regenerations. That is $K(t) := \max\{k \in \mathbb{N}_0 : \tau_k \leq t\}$. Here $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\tau_0 = 0$ by definition.

a) Provide the density function of τ_1. (4pt)

b) Argue that for $n \in \{0, 1, \cdots\}$

$$\mathbb{P}(K(t) \geq n) = \frac{2}{\sqrt{2\pi}} \int_{na/\sqrt{t}}^{\infty} e^{-x^2/2} dx.$$

(4pt)

c) For $x \in (-\infty, a)$, Argue that

$$\mathbb{P}(X(t) \leq x | \tau_1 > t) = \frac{\mathbb{P}(B(t) \in (x - 2a, x))}{\mathbb{P}(\tau_1 > t)}.$$

(4pt)

Problem 5: Simulation

Let $\{N(t); t \geq 0\}$ be a renewal process with gamma distributed interarrival times with mean 2 and variance 2. That is. the interarrival times have a Gamma distribution with rate parameter 1 and shape parameter 2. Let T be a given strictly positive constant.

a) Show that one can simulate the value of $N(T)$ as follows. Let $U_1, U_2, \cdots$ be independent and identically distributed random variables on the interval $(0,1)$, which are simulated when they are needed in the following algorithm.

- If $U_1 < e^{-T}$ set $N(T) = 0$.
- If for $k \in \{1, 2, \cdots\}$, we have $\prod_{j=1}^{k} U_j > e^{-T}$, check whether

  $$\prod_{j=1}^{k+1} U_j < e^{-T}.$$

  If yes, set $N(T) = \lfloor k/2 \rfloor$, where $\lfloor k/2 \rfloor$ is the largest integer which is less than or equal to $k/2$. If $\prod_{j=1}^{k+1} U_j > e^{-T}$ repeat this step, with k replaced by $k+1$.

(6pt)

Hint: Think of Poisson processes and the relationship between gamma distributions and independent exponential distributed random variables.

b) Assume that there is exactly one arrival in the interval $[0, T]$. Show that it is possible to simulate the arrival time as follows.

Let U_a and U_1, U_2 and U_3 be independent and identically distributed random variables on the interval $(0,1)$. If $U_a < \frac{3}{3+T}$, then the first arrival time is T times the maximum of U_1 and U_2. If $U_a > \frac{3}{3+T}$, then the first arrival time it T times the second largest value among U_1, U_2 and U_3. (6pt)