MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Instructor: Benjamin Nill

Combinatorics II (Online course)
June 3, 2014
9:00 - 14:00

SOLUTIONS

1. (1,5 points) Give the rook polynomial of a rectangular 7×9 ‘chessboard’.

No justification necessary, no simplification necessary.
Solution:

The rook polynomial is

$$\sum_{k=0}^{7} \binom{7}{k}\binom{9}{k} k! x^k.$$

2. (3 points)

(a) (2 points) Count the number of functions $f : \{1, \ldots, 30\} \to \{1, \ldots, 70\}$ such that
- the range of f contains the elements 1 and 2, and
- the range of f has at least eleven elements.

(b) (1 point) How many of the functions in (a) are injective?

Your answers must be as explicit as possible.

Solution:

$(j+2)!$

(a) For every subset S of $\{3, \ldots, 70\}$ with $j \geq 9$ elements exist $S(30, j+2)$ surjective functions $\{1, \ldots, 30\} \to \{1, 2\} \cup S$. Thus, the answer is

$$\sum_{j=9}^{68} \binom{68}{9}(j+2)!S(30, j+2) = \sum_{j=9}^{68} \binom{68}{9} \sum_{k=0}^{j+2} (-1)^k \binom{j+2}{j+2-k}(j+2-k)^{30}.$$

(b) In this case, we only pick subsets of size 28, so the answer is

$$\binom{68}{28} 30!.$$

3. (4 points) Find a closed formula for the sequence a_n that satisfies $a_0 = 17/4$, $a_1 = 31/4$, and for $n \geq 2$ the recursion relation

$$a_n - 9a_{n-2} = (-3)^{n+1} - 8n.$$

Clearly present every step of your computation.

Solution:

The characteristic roots are $3, -3$. The right side indicates to set up $a_n^{(p)} = An(-3)^{n+1} + Bn + C$ as a particular solution. This leads to $A = 1/2, B = 1, C = 9/4$. Solving $a_n = c_1 3^n + c_2(-3)^n + a^{(p)}$ for the initial conditions gives the solution

$$3^n + (-3)^n + \frac{n}{2}(-3)^{n+1} + n + \frac{9}{4}.$$

4. (3 points) Consider in this problem only connected graphs with 48 vertices and 47 edges.

Look at the following list of properties:

(1) bipartite
(2) planar
(3) has an Eulerian trail
(4) complete
(5) chromatic number 2
(6) has a Hamiltonian path

Decide for every property whether it

(A) holds for all such graphs
(S) holds for at least one such graph but not for all such graphs
(N) holds for no such graphs

For instance write, 1A or 1S or 1N, and 2A or 2S or 2N, etc.

You MUST shortly justify your answers.

Solution:

Note that any such graph must be a tree (connected, and has minimal number of possible edges). Hence, 1A (why?), 2A (again, why?), 3S (holds, if a path, but not otherwise), 4N (complete graphs with more than two vertices have cycles), 5A (bipartite), 6S (again holds if and only if it is a path).

5. (5 points) Let $k \geq 4$ be an integer, and $S = \{1, \ldots, k\}$. Let G_k be the graph

- whose vertices are the subsets of S of size 4,
- and where two vertices are adjacent, if the corresponding subsets of S intersect.

Correction: It should have been said that the graph shouldn't have loops, so that a subset is not adjacent to itself.

Note that G_k is connected. (You don't have to prove this).

(a) (1,5 points) Give an explicit numerical criterion (in terms of k) for deciding whether G_k has an Eulerian circuit.

Correction: It would have been better to have asked to prove that it has an Eulerian circuit.

(b) (2 points) Show that G_k is not planar for $k \geq 5$.

For full credits you must present TWO proofs that are as different as possible. Make an effort to give clear and readable arguments.

(c) (1,5 points) Let a_k be the number of vertices of G_k. Write the generating function for a_k as a quotient of two polynomials.

Solution:

(a) Given a subset of size 4 let $1 \leq j \leq 3$ be the number of elements in the intersection with another subset of size 4, then the number of these other subsets that intersect the given subset equals

$$\sum_{j=1}^{3}\binom{4}{j}\binom{k-4}{4-j}=4\binom{k-4}{3}+6\binom{k-4}{2}+4(k-4).$$

Since this number is always even, G_k has an Eulerian circuit.

(b) Proof 1: Use Kuratowski's theorem (it is easy to find a K_5 as a subgraph of G_k). Note that for $k=5$, G_k is isomorphic to K_5. Proof 2: Assume that G_k is planar (for $k \geq 6$, $k=5$ being direct) and deduce a contradiction from $2|E| = |V|(4\binom{k-4}{3}+6\binom{k-4}{2}+4(k-4))$ and $|E| \leq 3|V|-6$.

(c) $a_k = \binom{k}{4}$. Recall $\sum_{r=0}^{\infty}\binom{r+4}{4}x^r = \sum_{r=0}^{\infty}\binom{5+r-1}{r}x^r = \frac{1}{(1-x)^5}$. Multiplying by x^4 and reindexing we get

$$\sum_{k=0}^{\infty}\binom{k}{4}x^k = \frac{x^4}{(1-x)^5}.$$

6. (2 points) Let a_r be the number of nonnegative integer solutions of the equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 = r,$$

where $x_2 \leq 2$.

(a) (1 point) Mark all correct interpretations of a_{12} in terms of (unordered) partitions:

(A) a_{12} is the number of partitions of 12 into 4 parts, where the second smallest summand appears at most twice

(B) a_{12} is the number of partitions of 12 into 4 parts, where the second smallest summand is at most of size 2

(D) a_{12} is the number of partitions of 12 into 4 parts, where 2 occurs at most twice as a summand

(E) a_{12} is the number of partitions of 12 into at most 4 parts, where the second smallest summand appears at most twice

(F) a_{12} is the number of partitions of 12 into at most 4 parts, where the second smallest summand is at most of size 2

(G) a_{12} is the number of partitions of 12 into at most 4 parts, where 2 occurs at most twice as a summand

(H) a_{12} is the number of partitions of 12 into parts of size at most 4, where the second smallest summand appears at most twice

(I) a_{12} is the number of partitions of 12 into parts of size at most 4, where the second smallest summand is at most of size 2

(J) a_{12} is the number of partitions of 12 into parts of size at most 4, where 2 occurs at most twice as a summand

(K) a_{12} is the number of partitions of 12 with some part of maximal size 4, where the second smallest summand appears at most twice

(L) a_{12} is the number of partitions of 12 with some part of maximal size 4, where the second smallest summand is at most of size 2

(M) a_{12} is the number of partitions of 12 with some part of maximal size 4, where 2 occurs at most twice as a summand

No justifications necessary.

(b) (1 point) Write the generating function of the sequence a_r as a quotient of two polynomials.

Solution:

(a) The best answer is J (note that no summand must be necessarily equal to the maximal possible size 4). By the way, the Ferrers graph doesn't really help here (try it out).

(b) This is

$$\frac{1+x^2+x^4}{(1-x)(1-x^3)(1-x^4)}.$$

7. (2,5 points) In a supermarket one neatly puts 5 different types of apples on two shelfs, on each shelf precisely 20 apples sorted from the left to the right. If every type of apple must be used on each shelf, how many possibilities are there?

As usual it is enough to write down an explicit formula, you don't have to evaluate it.

Solution:

For one shelf the number of possibilities equals the coefficient of $\frac{x^{20}}{20!}$ in $(e^x-1)^5$, thus the binomial theorem yields

$$5^{20} - 5\cdot 4^{20} + 10\cdot 3^{20} - 10\cdot 2^{20} + 5.$$

By the rule of product the answer is $(5^{20} - 5\cdot 4^{20} + 10\cdot 3^{20} - 10\cdot 2^{20} + 5)^2$.

8. (2,5 points) How many outcomes are possible, if in a lottery 49 identical balls get sorted into 4 numbered urns where the machine is constructed in such a way that the first urn contains an odd number of balls and the last three urns each an even number of balls?

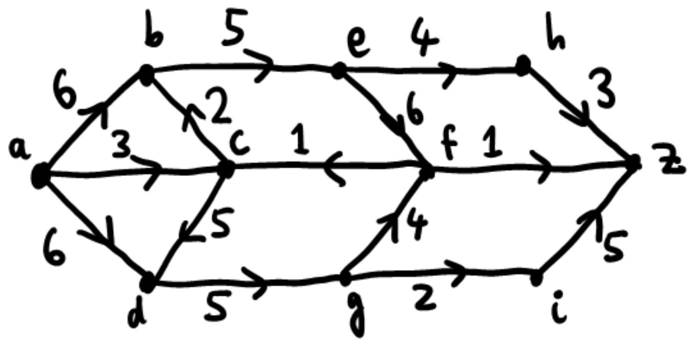
Solution:

We need the coefficient of x^{49} in

$$(x+x^3+x^5+\cdots)(1+x^2+x^4+\cdots)^3 = x(1+x^2+x^4+\cdots)^4 = \frac{x}{(1-x^2)^4}.$$

Since $\sum_{r=0}^{\infty}\binom{4+r-1}{r}(x^2)^r = \frac{1}{(1-x^2)^4}$, we get the right coefficient for $r=24$, so the answer is $\binom{27}{24} = \binom{27}{3}$.

9. (6,5 points) Consider the following network:



(a) (2,5 points) Use Dijkstra's algorithm to find for any vertex $v = a, b, c, d, e, f, g, h, i, z$ the distance $d(a, v)$ from a to v.

You MUST USE Dijkstra's algorithm in order to get any credits. Clearly show your table and in which order you choose the vertices (use an extra page, since you will need some space).

Solution:

Actually, $d(a, a) = 0$, so no need to have a column or row for a.

b	6	$\underline{5}$							
c	$\underline{3}$								
d	6	6	$\underline{6}$						
e	∞	∞	10	$\underline{10}$					
f	∞	∞	∞	∞	16	15	15	$\underline{15}$	
g	∞	∞	∞	11	$\underline{11}$				
h	∞	∞	∞	∞	14	14	$\underline{14}$		
i	∞	∞	∞	∞	∞	$\underline{13}$			
z	∞	∞	∞	∞	∞	∞	18	17	$\underline{16}$

The found distances to a are underlined.

(b) (1 points) Give a shortest directed path from a to z.

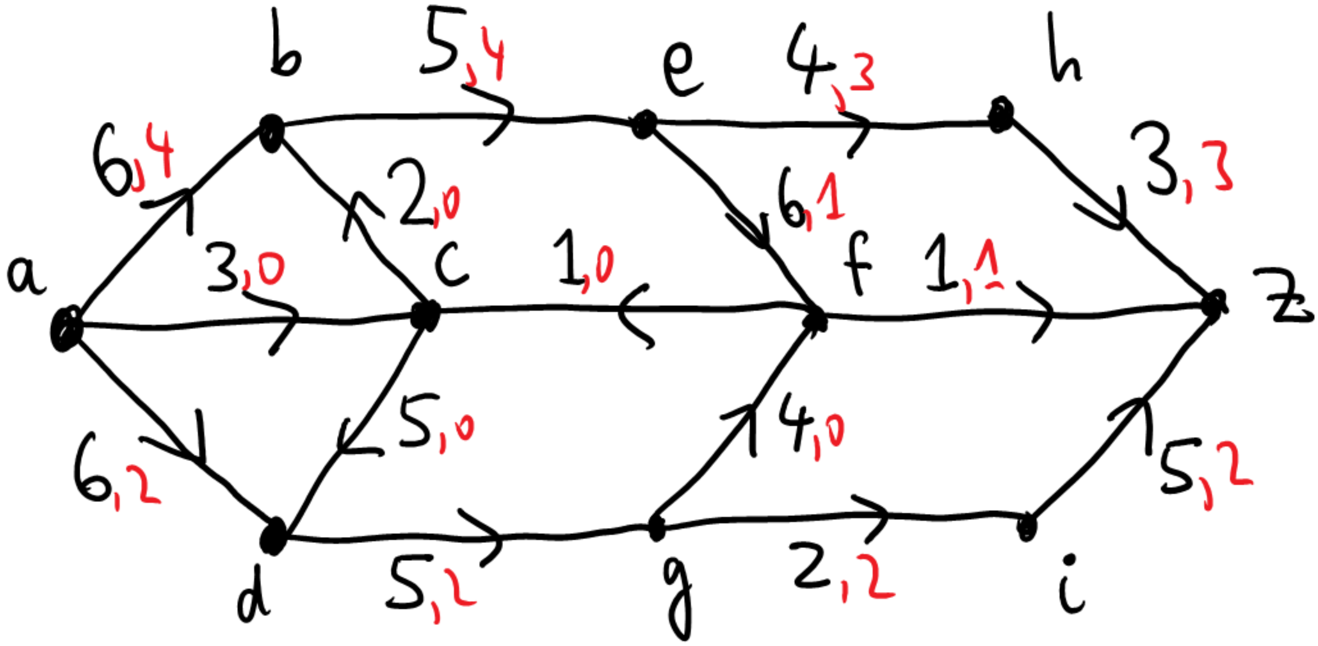
Solution:

$a \to d \to g \to f \to z.$

TURN PAGE!

(c) (2 points) Give a flow with maximal flow value. Enter the maximal flow you found RIGHT HERE into the network (next to the capacities):

Solution:



Write here the flow value of your flow: Solution: 6.

(d) (1 points) Give a cut with the minimal cut capacity RIGHT HERE:

Solution:

$P = \{a, b, c, d, e, f, g, h\}, \ \bar{P} = \{i, z\}$.

Write here the cut capacity of your cut: Solution: 6.